University of Kentucky, Physics 306
Homework #2, Rev. A, due Monday, 2022-01-30

1. Projections: relational versus parametric linear/planar geometry.

a) Show graphically that the following equations define the set of points $\{\boldsymbol{x}\}$ on a **line** or **plane**,

	relational	parametric
line	$\{\boldsymbol{x} \mid \boldsymbol{a}\times\boldsymbol{x}=\boldsymbol{d}\}$	$\{\boldsymbol{x}=\boldsymbol{x}_1+\boldsymbol{a}\alpha \mid \alpha\in\mathbb{R}\}$
plane	$\{\boldsymbol{x} \mid \boldsymbol{A}\cdot\boldsymbol{x}=D\}$	$\{\boldsymbol{x}=\boldsymbol{x}_2+\boldsymbol{b}\beta+\boldsymbol{c}\gamma \mid \beta,\gamma\in\mathbb{R}\}$

where $\boldsymbol{a},\boldsymbol{d},\boldsymbol{A},D$ are constants which define the geometry, $\boldsymbol{x}_{1,2}$ are fixed points on the line and plane respectively, and α,β,γ are parameters that vary along the line/plane (they uniquely parametrize points in the line/plane). [bonus: Show the 5th relation $\{\boldsymbol{x}=\boldsymbol{x}_2+\boldsymbol{A}\times\boldsymbol{\delta} \mid \boldsymbol{\delta}\in\mathbb{R}^3\}$.]

b) What constraint between $\boldsymbol{a}$ and $\boldsymbol{d}$ is implicit in the formula $\boldsymbol{a}\times\boldsymbol{x}=\boldsymbol{d}$? What is the relation between $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{A}$? Substitute $\boldsymbol{x}$ of each parametrization into its relational equation to show the consistency between the two forms and to derive $\boldsymbol{d}$ and D in terms of $\boldsymbol{a},\boldsymbol{x}_1$ and $\boldsymbol{A},\boldsymbol{x}_2$.

c) Define $\tilde{\boldsymbol{a}}\equiv\boldsymbol{A}/(\boldsymbol{a}\cdot\boldsymbol{A})$, which is parallel to $\boldsymbol{A}$, and *normalized* in the sense that $\boldsymbol{a}\cdot\tilde{\boldsymbol{a}}=1$. Using the BAC-CAB rule, show that $\boldsymbol{x}=\boldsymbol{a}(\tilde{\boldsymbol{a}}\cdot\boldsymbol{x})-\tilde{\boldsymbol{a}}\times(\boldsymbol{a}\times\boldsymbol{x})$ for all $\boldsymbol{x}$. Illustrate this non-orthogonal projection of $\boldsymbol{x}$ onto vectors $\boldsymbol{x}_1$ parallel to the line plus $\boldsymbol{x}_2$ parallel to the plane. [bonus: use this to calculate the point $\boldsymbol{x}_0$ at the intersection of the line and plane in terms of $\boldsymbol{a},\boldsymbol{d},\boldsymbol{A},D$. Verify this by showing that $\boldsymbol{x}_0$ satisfies the relational equation for both the line and plane.]

d) Let $\tilde{\boldsymbol{a}}\equiv\frac{\vec{\boldsymbol{b}}\times\vec{\boldsymbol{c}}}{\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}}\times\vec{\boldsymbol{c}}}$, $\tilde{\boldsymbol{b}}\equiv\frac{\vec{\boldsymbol{c}}\times\vec{\boldsymbol{a}}}{\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}}\times\vec{\boldsymbol{c}}}$, and $\tilde{\boldsymbol{c}}\equiv\frac{\vec{\boldsymbol{a}}\times\vec{\boldsymbol{b}}}{\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}}\times\vec{\boldsymbol{c}}}$, where the arrows on $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ distinguish them from their *covectors* $\tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}},\tilde{\boldsymbol{c}}$. The definition of $\tilde{\boldsymbol{a}}$ is the same as in c) with $\boldsymbol{A}=\vec{\boldsymbol{b}}\times\vec{\boldsymbol{c}}$. Calculate the nine combinations of $(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}},\vec{\boldsymbol{c}})^T\cdot(\tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}},\tilde{\boldsymbol{c}})=I$, i.e. $\vec{\boldsymbol{a}}\cdot\tilde{\boldsymbol{a}}=1$, $\vec{\boldsymbol{a}}\cdot\tilde{\boldsymbol{b}}=0,\ldots$ to show they are *mutually orthonormal*. In this sense, $(\tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}},\tilde{\boldsymbol{c}})$ is the *dual* basis of $(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}},\vec{\boldsymbol{c}})$. [bonus: Show using Cramer's rule that $[\tilde{\boldsymbol{a}}|\tilde{\boldsymbol{b}}|\tilde{\boldsymbol{c}}]=[\vec{\boldsymbol{a}}|\vec{\boldsymbol{b}}|\vec{\boldsymbol{c}}]^{-1}$, ie. that it is a *reciprocal basis*.]

e) The *contravariant components* of $\boldsymbol{x}$ are defined as the components (α,β,γ) that satisfy the equation $\boldsymbol{x}=\vec{\boldsymbol{a}}\alpha+\vec{\boldsymbol{b}}\beta+\vec{\boldsymbol{c}}\gamma$; i.e., $(\vec{\boldsymbol{a}},\vec{\boldsymbol{b}},\vec{\boldsymbol{c}})$ is the contravariant basis. Using $\boldsymbol{x}\cdot\tilde{\boldsymbol{a}}$, etc., calculate the three contravariant components of $\boldsymbol{x}$ in terms of dot products. Likewise, find the *covariant components* $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})$ of $\boldsymbol{x}$, defined as the components $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma})$ which satisfy $\boldsymbol{x}=\tilde{\boldsymbol{a}}\tilde{\alpha}+\tilde{\boldsymbol{b}}\tilde{\beta}+\tilde{\boldsymbol{c}}\tilde{\gamma}$; i.e. $(\tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}},\tilde{\boldsymbol{c}})$ is the covariant basis.

2. The **Pauli matrices**, defined as $\sigma_x\equiv\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_y\equiv\begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\sigma_z\equiv\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, are used to describe quantum mechanical spin-1/2 particles. Show that $\sigma_j\sigma_k=\delta_{jk}I+i\varepsilon_{jk\ell}\sigma_\ell$. What is the *triple product* $\sigma_x\sigma_y\sigma_z$? Thus σ_j acts like a basis vector, the product of the Pauli matrices has the structure of the dot, cross, and triple products.

[bonus: Show that product of *Dirac matrices* is $\gamma^\mu\gamma^\nu=g^{\mu\nu}I+i\sigma^{\mu\nu}$ and calculate the components $g^{\mu\nu}$ and the matrices $\sigma^{\mu\nu}$.]