University of Kentucky, Physics 306
Homework #3, Rev. A, due Monday, 2023-02-06

1. Rotations of the vector $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ are generated by the matrix $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which represents a 90° CCW rotation—just as complex rotations $e^{i\phi}$ are generated by i, where $i^2 = -1$.

a) Show that $M\boldsymbol{v}$ rotates $\boldsymbol{v}$ by 90° CCW, and that $M^2 = -I$.

b) Show that $R = e^{M\phi} = I\cos\phi + M\sin\phi$ and calculate the components of R. Show that $M^T = -M$ (the generator is asymmetric) and thus $R^T R = I$, just like $(e^{i\phi})^*(e^{i\phi}) = 1$. Show that $dR = MRd\phi$, just like $de^{i\phi} = ie^{i\phi}d\phi$ in H02.

c) The cross product $\times$ generates rotation in 3d. Calculate the matrices $\boldsymbol{M} = (M_x, M_y, M_z)$, where $\hat{\boldsymbol{x}} \times \boldsymbol{v} = M_x \boldsymbol{v}$, etc. Show that the components of M_ℓ are $(M_\ell)_{jk} = \varepsilon_{kj\ell}$ and that $M_\ell^2 = -P_{\perp\ell}$, where $P_{\perp\ell}$ projects perpendicular to $\hat{\boldsymbol{e}}_\ell$. Thus the general matrix for a CCW rotation of angle v about the $\hat{\boldsymbol{v}}$-axis is $R_{\boldsymbol{v}} = e^{\boldsymbol{M}\cdot\boldsymbol{v}} = I\cos v + \boldsymbol{M}\cdot\hat{\boldsymbol{v}}\sin v + \hat{\boldsymbol{v}}\hat{\boldsymbol{v}}^T(1-\cos v)$, the *Rodrigues' formula*. The third term projects out the axis of rotation $\hat{\boldsymbol{v}}$. Verify this formula for the familiar case $\vec{\boldsymbol{v}} = \hat{\boldsymbol{z}}\phi$.

d) [bonus: The Pauli matrices also generate rotation. Calculate $e^{-i\sigma_x\phi}$, $e^{-i\sigma_y\phi}$, $e^{-i\sigma_z\phi}$.]

2. Minkowski space. We will derive the Lorentz transformations using matrices from Einstein's principle of special relativity that the speed of light c is constant in any reference frame.

a) Show that the principle of relativity can be written as $\boldsymbol{x}^2 = \boldsymbol{x}\cdot\boldsymbol{x} = \boldsymbol{x}^T g\boldsymbol{x} = 0$ for the *space-time vector* $\boldsymbol{x} = (ct\ x)^T$ with components representing the distance $x = ct$ travelled by a photon in time t, where the matrix $g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the *Minkowski metric*. In three-dimensional space, $\boldsymbol{x} = (ct\ x\ y\ z)^T$ is a 4-vector, and the metric becomes a 4×4 matrix $g = \mathrm{diag}(-1, 1, 1, 1)$.

b) Momentum and energy can also be combined into the space-time vector $\boldsymbol{p} = (E/c\ p)^T$. Show that invariance of the dot product $\boldsymbol{p}^T g\boldsymbol{p} = -(mc)^2$ leads to the formula $E^2 = (pc)^2 + (mc^2)^2$, which reduces to Einstein's equation $E = mc^2$ when $p = 0$.

c) Normal rotations R keep the length of vector constant by preserving the Euclidian metric I: $R^T IR = I$. Likewise, Lorentz transformations Λ preserve the Minkowski metric: $\Lambda^T g\Lambda = g$. For a small 'rotation' $\Lambda = I + Gd\alpha$ generated by G, show that $G^T g + gG = 0$. Show that $G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ satisfies this relation to first order in $d\alpha$. Note the difference between G and M!

d) Show that $G^2 = I$ and therefore $\Lambda = e^{G\alpha} = I\cosh\alpha + G\sinh\alpha = \gamma\begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}$, where $\beta = \tanh\alpha = v/c$ and $\gamma = \cosh\alpha = (1-\beta^2)^{-1/2}$. Thus the Lorentz transformations $\boldsymbol{x}' = \Lambda\boldsymbol{x}$ are $t' = \gamma(t + vx/c^2)$ and $x' = \gamma(x + vt)$, which encode all of the features of special relativity. Verify that $\Lambda^T g\Lambda = g$ and plot the rotated basis vectors. Calculate the relativisitic addition rule for velocities β_1 and β_2 from $\Lambda = \Lambda_2\Lambda_1$.

e) [bonus: the Dirac matrices γ^μ generate Lorentz rotations. Calculate $e^{-i\gamma_\mu v^\mu}$.]