University of Kentucky, Physics 306
Homework #4, Rev. B, due Monday, 2023-02-13

1. Similarity transforms and invariants of the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ and in general.

a) What are the eigenvectors and eigenvalues of σ_z? Diagonalize σ_x and σ_y to obtain their eigenbasis $U_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $U_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$. Verify that U_x and U_y are unitary.

b) Justify the eigenequations $\sigma_x U_x = U_x \sigma_z$ and $\sigma_y U_y = U_y \sigma_z$. For $i, j{=}x, y, z$, find the matrices U_{ij} of the *similarity transforms* $\sigma_i = U_{ij} \sigma_j U_{ij}^\dagger$ (the $_i$, $_j$, and $_{ij}$ are not component indices!).

c) [bonus: The *trace* of a matrix is the sum of its diagonal elements: $\text{tr}A = A_{ii}$. Calculate $\text{tr}(0)$ and $\text{tr}(\sigma_i)$. Show that $\text{tr}(ABC) = \text{tr}(BCA) = \text{tr}(CAB)$ but not necessarily $\text{tr}(ABC) = \text{tr}(CBA)$. Thus if $A' = UAU^\dagger$ then $\text{tr}(A') = \text{tr}(A)$, i.e. the trace, or 'perimeter', of A is invariant.]

d) [bonus: The *determinant* of a matrix is the fully antisymmetric product of one element in each row or column: $\det A = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k} = \varepsilon_{ijk} A_{i1} A_{j2} A_{k3}$. Note that $\det(A^\dagger) = \det(A)^*$ and $\det(AB) = \det(A)\det(B)$. Calculate $\det(I)$ and $\det(\sigma_i)$. Show that if $U^\dagger U = I$ then $|\det(U)| = 1$, and if $A' = UAU^\dagger$, then $\det(A') = \det(A)$, i.e. the determinant, or 'volume', of A is invariant. Show that $\exp(\text{tr}(A)) = \det(\exp(A))$, for example, $\exp(\text{tr}(0)) = 1 = \det(\exp(0) = I)$.]

e) [bonus: Show that the *characteristic equation* $|A - \lambda I| = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$, substituting $\lambda \to A$ in the second equality, is invariant under similarity transforms. Thus A has n independent invariants. The first and last coefficients a_{n-1} and a_0 are $\text{tr}(A)$ and $\det(A)$, respectively, while the others are k-dimensional 'perimeters', for example, *surface area*, of the transformation.]

f) [bonus: Prove the Cayley-Hamilton Theorem (that any matrix A satisfies its own characteristic equation) for the case of diagonal matrices. The full theorem follows from the invariance of the characteristic equation. Show that $A^{-1} = -(A^{n-1} + a_{n-1}A^{n-2} + \ldots + a_1 I)/a_0$.]

2. Stretches. In analogy with the *polar decomposition* $w = x + iy = \rho e^{i\phi}$ of complex numbers, any matrix A can be decomposed $A = RS$ into a stretch S and a rotation R, which are the building blocks of all linear operators. This problem explores the structure of stretches.

a) A symmetric matrix S is guaranteed to have a complete set of eigenvectors $V = (\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots)$ and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots$, such that $S\boldsymbol{v}_i = \boldsymbol{v}_i \lambda_i$. Augment these n equations to obtain $SV = VW$, where $W = \text{diag}(\lambda_1, \lambda_2, \ldots)$, and thus show $S = VWV^\dagger$ and $W = V^\dagger SV$.

b) Show that the eigenvalues of S must be real: $\lambda^* = \lambda$, and that two eigenvectors $\boldsymbol{v}_i, \boldsymbol{v}_j$ of S with distinct eigenvalues $\lambda_i \neq \lambda_j$ must be orthogonal: $\boldsymbol{v}_i \cdot \boldsymbol{v}_j = 0$, and thus show the matrix of eigenvectors is unitary: $V^\dagger V = I$. [bonus: Interpret the similarity transforms geometrically.]

c) Calculate the eigenvalues and eigenvectors of $M_z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ Do they look familiar? Show that $M_z = VWV^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ and $e^{M_z\phi} = Ve^{W\phi}V^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$. Multiply this out to verify H03#1b). This is an example of the *normal matrix analogy*.