

University of Kentucky, Physics 306
Homework #6, Rev. A, due Monday, 2023-02-27

1. The **Fourier Series** is an application of using an orthogonal basis to extract components of a generalized vector (function). Orthogonality is crucial, because otherwise it would be impossible to solve the linear system of equations (invert an infinite-dimensional matrix) to obtain each component.

a) Submit a screenshot of your completed Level 3 in the **Fourier: Making Waves** Wave Game.

b) Calculate the inner products $\langle n|m \rangle = \int_{-\pi}^{\pi} d\phi \Phi_n^*(\phi) \Phi_m(\phi)$ of the Fourier sine basis functions $\Phi_n(\phi) = \frac{1}{\sqrt{\pi}} \sin(n\phi)$. Use this orthonormal basis to calculate the components b_n of the Fourier sine series $f(\phi) = \sum_{n=1}^{\infty} b_n \Phi_n(\phi)$ of the function $f(\phi) = 1$ for $0 < \phi < \pi$ and $f(\phi) = -1$ for $-\pi < \phi < 0$. Plot the first 1, 2, 3, 5, 10 terms in Mathematica. What does the periodic series look like outside the interval $-\pi < \phi < \pi$? Compare the inner product $\langle f|f \rangle = \int_{-\pi}^{\pi} d\phi f^*(\phi) f(\phi)$ of $f(\phi)$ with that of its Fourier components $\langle f|f \rangle = \sum_n b_n^* b_n$. [it's a **rotation** from the $|\phi\rangle$ to $|n\rangle$ basis!]

2. The **Legendre Polynomials** $P_\ell(x)$ appear in the solution of Laplace's equation in spherical coordinates. They can be defined as the series of orthogonal polynomials of degree $\ell = 0, 1, 2, \dots$ with respect to the weight $w(x) = 1$ over the range $-1 \leq x \leq 1$, *standardized* to $P_\ell(1) = 1$.

a) Show that $\langle x^m|x^n \rangle = \frac{2}{m+n+1}$ [$m+n$ is even] or 0 [$m+n$ is odd]. Apply the **Gram-Schmidt** procedure to the basis of monomials $1, x, x^2, \dots$ to generate the series of $P_\ell(x)$ up to order $\ell = 4$. Calculate the *normalization* $h_\ell = \langle P_\ell|P_\ell \rangle$. Expand $f(x) = x^4$ in the Legendre basis.

b) Show that these basis functions $P_\ell(x)$ are eigenfunctions of the linear Legendre differential operator $L = -\frac{d}{dx}(1-x^2)\frac{d}{dx}$ by direct calculation of their eigenvalues $L[P_\ell(x)] = \lambda P_\ell(x)$.

3. The **Sturm-Liouville** 2nd order linear differential operator, which generalizes 2b) to

$$L[y(x)] \equiv \frac{1}{w(x)} \left[\frac{d}{dx} p(x) \frac{d}{dx} - q(x) \right] y(x), \quad (1)$$

is self-adjoint, $L^\dagger = L$, [it's a **stretch**!] with respect to the inner product

$$\langle y_1|y_2 \rangle \equiv \int_a^b w(x) dx y_1^*(x) y_2(x) \quad (2)$$

if we impose the boundary conditions $y(a)w(a) = y(b)w(b) = 0$. Thus L has real eigenvalues λ_i and a complete set of orthogonal eigenfunctions. Any function $f(x)$ has the expansion $|f\rangle = \sum_i |u_i\rangle f_i$ or $f(x) = \sum_i u_i(x) f_i$, with components $f_i = \langle u_i|f \rangle / \langle u_i|u_i \rangle = \int_a^b w(x) dx u_i^*(x) f(x) / h_i$.

a) Show that L is *self-adjoint* or Hermitian. *Hint*: use the definition $\langle f|H^\dagger g \rangle \equiv \langle Hf|g \rangle$ to show that the operator $D = \frac{1}{w} \frac{d}{dx}$ is anti-Hermitian and apply it to the composition of operators in L . [bonus: Show that any linear 2nd-order differential operator over any weight can be put in this form.]

b) Given eigenfunctions $L|u_i\rangle = \lambda_i|u_i\rangle$, show that $\lambda_i \in \mathbb{R}$ and that $\langle u_i|u_j \rangle = 0$ if $\lambda_i \neq \lambda_j$. Note that completeness is much harder to prove. Operate L on the expansion of $|f\rangle$ in the basis $|u_i\rangle$ to show that its *spectral decomposition* is $L = \sum_i \lambda_i |u_i\rangle \langle u_i|$. Do the same for the identity $I|f\rangle = |f\rangle$.

c) Compile a chart of $w(x), a, b, p(x), q(x), \lambda_i, h_i$ for each of these *orthogonal function bases*:

- i) cylindrical harmonics $e^{im\phi}$; ii) associated Legendre functions $P_\ell^{(m)}(x)$; iii) Fourier series $\sin(k_n x)$;
- iv) Bessel functions $J_m(k_n x)$; v) spherical Bessel functions $j_\ell(k_n x)$; vi) Hermite polynomials $H_n(x)$;
- vii) associated Laguerre polynomials $L_n^{(\alpha)}(x)$. [see NIST [Digital Library of Mathematical Functions](#)]