University of Kentucky, Physics 306
Homework #7, Rev. A, due Monday, 2023-03-20

1. Vectors in **curvilinear coordinates** (q^1, q^2, q^3) have a natural *coordinate basis* $\vec{\boldsymbol{b}}_i \equiv \partial\vec{\boldsymbol{r}}/\partial q^i$ and *reciprocal basis* $\vec{\boldsymbol{b}}^i \equiv \boldsymbol{\nabla} q^i = \partial q^i/\partial\vec{\boldsymbol{r}}$. Note that each basis vector is actually a *vector field* (a function of position). The most common coordinate systems are Cartesian $q^i = (x, y, z)$, cylindrical $q^i = (\rho, \phi, z)$, and spherical $q^i = (r, \theta, \phi)$, defined by the transformations $x + iy = \rho e^{i\phi}$ and $z + i\rho = re^{i\theta}$, respectively. These are all orthogonal, right-handed systems, for which both bases are aligned with the *orthonormal basis* $\hat{\boldsymbol{e}}_i = \vec{\boldsymbol{b}}_i/h_i = \vec{\boldsymbol{b}}^i h_i$, where $\vec{\boldsymbol{b}}_i = \partial\vec{\boldsymbol{r}}/\partial q^i$ is the unnormalized *coordinate basis* and $h_i = |\vec{\boldsymbol{b}}_i| = 1/|\vec{\boldsymbol{b}}^i|$ is called the *scale factor*.

a) Determine the coordinate transformation $q^i(q^{i'})$ from each coordinate system to each of the others. *Hint: invert and combine the two transformations above.*

b) For each coordinate system, illustrate the three coordinate isosurfaces $q^i(\vec{\boldsymbol{r}}) = q^i_0$ (constant) passing through an abritrary point $\vec{\boldsymbol{r}}_0$, labeling lengths and angles in your diagram. For each coordinate q^i, identify the curve $\vec{\boldsymbol{s}}(q^i; q^j_0, q^k_0)$ at the intersection of two surfaces of constant $q^j = q^j_0$ and $q^k = q^k_0$.

c) For cylindrical and spherical coordinates, calculate $\vec{\boldsymbol{b}}_i = \partial\vec{\boldsymbol{r}}/\partial q^i$ using $\vec{\boldsymbol{dr}} = \hat{\boldsymbol{x}}dx + \hat{\boldsymbol{y}}dy + \hat{\boldsymbol{z}}dz$. Normalize $\vec{\boldsymbol{b}}_i = \hat{\boldsymbol{e}}_i h_i$ to find the *unit vectors*. The scale factors h_θ and h_ϕ for angular coordinates are just the radii of curvature, according to the arc length formulae $ds_\theta = rd\theta$ and $ds_\phi = \rho d\phi$.

d) Construct the transformation matrices between unit bases, by considering rotations $R_z(\phi)$ (rotation by an angle ϕ about the z-axis) and $R_\phi(\theta)$ (about the y-axis). Compare with part c).

e) For each coordinate system, calculate the *line element* $\vec{\boldsymbol{dl}} = \hat{\boldsymbol{e}}_i h_i dq^i$, the *area element* $\vec{\boldsymbol{da}} = \frac{1}{2}\vec{\boldsymbol{dl}} \times \vec{\boldsymbol{dl}} = \hat{\boldsymbol{e}}_k\, h_i h_j\, dq^i dq^j$, and the *volume element* $d\tau = \frac{1}{3}\vec{\boldsymbol{dl}} \cdot \vec{\boldsymbol{da}} = h_1 h_2 h_3\, dq^1 dq^2 dq^3$.

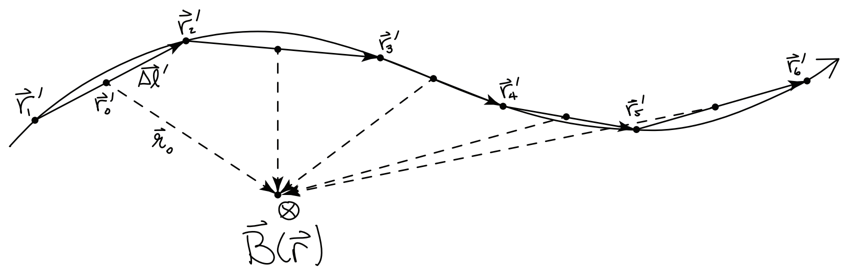
f) For each coordinate system, calculate the *metric* $g_{ij} = \boldsymbol{b}_i \cdot \boldsymbol{b}_j = \mathrm{diag}(h_1^2, h_2^2, h_3^2)$.

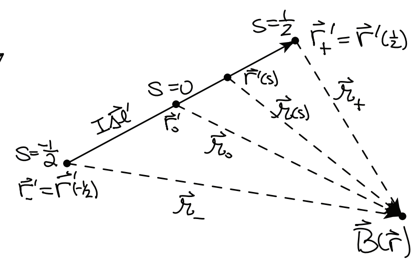
g) Invert the cylindrical coordinate functions to obtain $(\rho, \phi, z) = f^{-1}(x, y, z)$. Calculate the covariant basis $\boldsymbol{b}^i = \nabla q^i = \hat{\boldsymbol{e}}_i/h_i$ and verify that $\boldsymbol{b}_i \cdot \boldsymbol{b}^j = \delta^j_i$. Calculate $g^{ij} = \boldsymbol{b}^i \cdot \boldsymbol{b}^j = \mathrm{diag}(h_1^2, h_2^2, h_3^2)$. [bonus: Do the same for spherical coordinates.]

2. The magnetic analog of *Coulomb's law* (with a scalar charge element $dq = \lambda dl = \sigma da = \rho d\tau$) is the **Biot-Savart law** (with a vector current element $\boldsymbol{v}dq = I\vec{\boldsymbol{dl}} = \vec{\boldsymbol{K}}da = \vec{\boldsymbol{J}}d\tau$):

$$\boldsymbol{B} \;=\; \frac{\mu_0}{4\pi}\oint' \frac{\boldsymbol{v}dq' \times \boldsymbol{\mathscr{r}}}{\mathscr{r}^3} \;\approx\; \sum_i \left(\Delta\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{I\Delta\boldsymbol{\ell} \times \boldsymbol{\mathscr{r}}_0}{\mathscr{r}_0^3}\right)_i, \tag{1}$$

where $\Delta\boldsymbol{\ell}$ is the displacement vector from the beginning to the end of each current segment, and $\boldsymbol{\mathscr{r}}_0$ is the displacement vector from the middle of each current segment $\boldsymbol{r}'_0$ to the field point $\boldsymbol{r}$. The approximation is that all of the current is concentrated at $\boldsymbol{r}'_0$ instead of spread out along the length of the segment from $\boldsymbol{r}'_0 - \Delta\boldsymbol{\ell}/2$ to $\boldsymbol{r}'_0 + \Delta\boldsymbol{\ell}/2$. In this problem we first calculate a correction term to account for this difference, and then calculate the exact B-field due to each straight segment.





a) To analytically integrate the Biot-Savart law along a single straight segment of the path, parametrize the segment $\boldsymbol{r}'(s)$ with the parameter s, ranging from $s = -\frac{1}{2}$ at the beginning to $s = +\frac{1}{2}$ at the end of the segment. The parametrization involves the constant vectors $\boldsymbol{r}'_0$ (the center of the segment) and $\Delta\boldsymbol{\ell}$ (displacement along the segment). Calculate the line element $\boldsymbol{dl} = \frac{d\boldsymbol{r}'}{ds}ds$. Calculate $\boldsymbol{\mathscr{r}}$ as a function of $\boldsymbol{\mathscr{r}}_0$, $\Delta\boldsymbol{\ell}$, and s. Substitute these into the Biot-Savart formula and factor out the constant approximation of Eq. 1] to obtain

$$\Delta\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu 0}{4\pi}\frac{I\Delta\boldsymbol{\ell}\times\boldsymbol{\mathscr{r}}_0}{\mathscr{r}_0^3}T(\alpha,\beta), \tag{2}$$

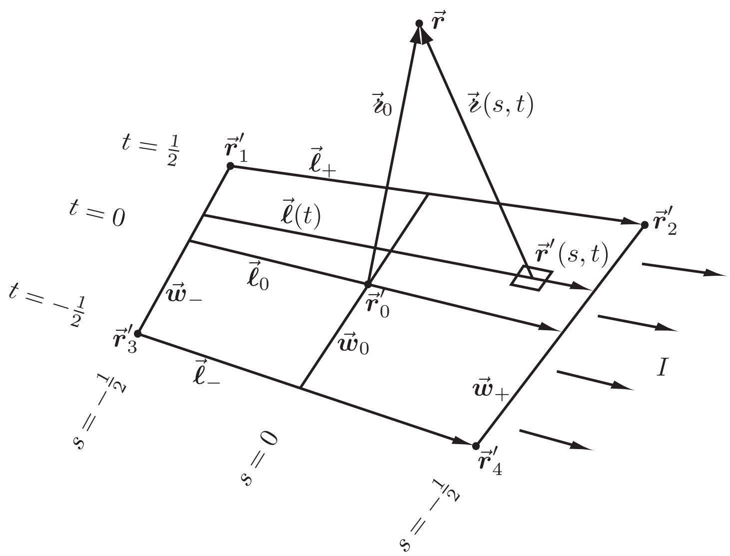
where the integral $T(\alpha,\beta)$ along s depends on $\alpha = \boldsymbol{\mathscr{r}}_0\cdot\Delta\boldsymbol{\ell}/\mathscr{r}_0^2$ and $\beta = \Delta\ell^2/\mathscr{r}_0^2$.

b) [bonus: Approximate the integrand of $T(\alpha,\beta)$ to order s^2 and integrate to obtain the correction term $T(\alpha,\beta) \approx 1+\frac{1}{8}(5\alpha^2-\beta)$ for the case where all the current is at the center of the segment.]

c) [bonus: Calculate the exact integral $T(\alpha,\beta)$ and show that $\Delta\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\frac{\Delta\boldsymbol{\ell}\times\boldsymbol{\mathscr{r}}_0}{(\Delta\boldsymbol{\ell}\times\boldsymbol{\mathscr{r}}_0)^2}\,\Delta\boldsymbol{\ell}\cdot(\hat{\boldsymbol{\mathscr{r}}}_- - \hat{\boldsymbol{\mathscr{r}}}_+)$, where $\boldsymbol{\mathscr{r}}_\pm = \boldsymbol{r} - \boldsymbol{r}'(\pm\frac{1}{2})$ is the displacement vector from each end of the segment to the field point and $\hat{\boldsymbol{\mathscr{r}}}_\pm = \boldsymbol{\mathscr{r}}_\pm/\mathscr{r}_\pm$.]

d) [bonus: show that this is equivalent to $\Delta\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\frac{(\boldsymbol{\mathscr{r}}_-\times\boldsymbol{\mathscr{r}}_+)(\mathscr{r}_-+\mathscr{r}_+)}{\mathscr{r}_-\mathscr{r}_+(\mathscr{r}_-\mathscr{r}_++\boldsymbol{\mathscr{r}}_-\cdot\boldsymbol{\mathscr{r}}_+)}$.]

3. Current sheet—surface currents can be approximated numerically by a tiling of quadrilaterals like the one shown below, with current I flowing parallel to the top and bottom edges, from left to right. Let the vector $\boldsymbol{\ell} = \boldsymbol{\ell}_+ = \boldsymbol{\ell}_-$ run along either the top or bottom edge, parallel to the current; $\boldsymbol{w} = \boldsymbol{w}_- = \boldsymbol{w}_+$ from bottom to top along the left or right edge; and $\boldsymbol{r}_0$ be the point at the center of the parallelogram as shown in the diagram.



a) Parametrize the surface of the parallelogram as $\boldsymbol{r}'(s,t)$, with the top and bottom edges are at $t = +\frac{1}{2}$ and $-\frac{1}{2}$, and the left and right edges are at $s = +\frac{1}{2}$ and $-\frac{1}{2}$ respectively.

b) Write down the Biot-Savart integral for the magnetic field in terms of integration parameters s, t and constants $\boldsymbol{\mathscr{r}}_0 \equiv \boldsymbol{r} - \boldsymbol{r}'_0$, $\boldsymbol{\ell}$, and $\boldsymbol{w}$.

c) [bonus: Expand in powers of s and t, to calculate the integral up to second order.]

d) [bonus: It is not possible to tile arbitrary surfaces with parallelograms—we need all four points on the quadrilateral to be arbitrary. To generalize this solution, let $\boldsymbol{\ell}_0$, $\boldsymbol{w}_0$ be the corresponding vectors through the center of the quadrilateral. We need one more vector $\boldsymbol{u}_0 = \boldsymbol{\ell}_+ - \boldsymbol{\ell}_- = \boldsymbol{w}_+ - \boldsymbol{w}_-$, where $\boldsymbol{\ell}_\pm$ run across the top and bottom, and $\boldsymbol{w}_\pm$ run along the right and left sides of the diagram. Generalize steps (a)-(c) to calculate $\boldsymbol{B}(\boldsymbol{r})$.]