University of Kentucky, Physics 306
Homework #8, Rev. A, due Monday, 2023-04-03

1. The **gradient**, **curl**, and **divergence** are all applications of d in different dimensions. They are defined as $df = \boldsymbol{\nabla} f \cdot \boldsymbol{dl}$, $d(\boldsymbol{A}\cdot\boldsymbol{dl}) = (\boldsymbol{\nabla}\times\boldsymbol{A})\cdot\boldsymbol{da}$, $d(\boldsymbol{B}\cdot\boldsymbol{da}) = (\boldsymbol{\nabla}\cdot\boldsymbol{B})\cdot d\tau$, respectively, in terms of the differential operator $d = dq^i\partial_i$ and elements $\boldsymbol{dl} = \hat{\boldsymbol{e}}_i h_i dq^i$, $\boldsymbol{da} = \frac{1}{2}\boldsymbol{dl}\times\boldsymbol{dl}$, $d\tau = \frac{1}{3}\boldsymbol{dl}\cdot\boldsymbol{da}$.

a) Apply these definitions to scalar f, polar vector $\boldsymbol{A}$, and axial vector $\boldsymbol{B}$ fields to obtain $\boldsymbol{\nabla} f = \frac{df}{\boldsymbol{dr}} = \frac{\hat{\boldsymbol{e}}_i}{h_i}\frac{\partial}{\partial q^i} f$, $\quad \boldsymbol{\nabla}\times\boldsymbol{A} = \frac{d\,\boldsymbol{dr}\cdot}{\boldsymbol{da}}\boldsymbol{A} = \frac{\hat{\boldsymbol{e}}_i}{h_j h_k}\frac{\partial}{\partial q^j} h_k A_k$, $\quad \boldsymbol{\nabla}\cdot\boldsymbol{B} = \frac{d\,\boldsymbol{da}\cdot}{\boldsymbol{d\tau}}\boldsymbol{B} = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial q^k} h_i h_j B_k$, where i, j, k are cyclic and $\frac{1}{\boldsymbol{dr}}$ is the inverse transformation of $\boldsymbol{dr}\cdot$ from $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$ to $\{dx, dy, dz\}$, etc.

b) Expand a) in Cartesian, cylindrical, and spherical coordinates (compare with this table).

c) Calculate the gradient of $f(r,\theta,\phi) = r^\ell P_\ell(\cos(\theta))$, using $\frac{x^2-1}{\ell}P'_\ell(x) = xP_\ell(x) - P_{\ell-1}(x)$,

d) the curl of $\boldsymbol{A}(x,y,z) = (\hat{\boldsymbol{y}}x - \hat{\boldsymbol{x}}y)/(x^2+y^2)^n$ in Cartesian and cylindrical coordinates, and the

e) divergence of $\boldsymbol{B}(x,y,z) = (\hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y + \hat{\boldsymbol{z}}z)/(x^2+y^2+z^2)^n$ in Cartesian and spherical coordinates.

f) Derive the formulas for $d^2 f$ and $d^2\boldsymbol{A}\cdot\boldsymbol{dl}$ in terms of their partial derivatives to show that the trivial 2^{nd} derivatives $\boldsymbol{\nabla}\times\boldsymbol{\nabla} f$ and $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times\boldsymbol{A}$ are special cases of $d^2 = 0$.

2. Potential theory—the *Fundamental Theorem of Differentials (FTD)* $d\int_r \omega + \int_r d\omega = \omega$ implies that if $d\omega = 0$, then the *potential* $\alpha \equiv \int_r \omega \equiv \left[\int_{r'=0}^{r} \omega_{r\Omega}(r', \Omega)dr'\right] d\Omega$ (integrated along radial coordinate lines) is its *antiderivative*: $d\alpha = \omega$. We will use this formula to derive the antiderivative of the following special vector fields, generalizing the *Fundamental Theorem of Calculus (FTC)*:

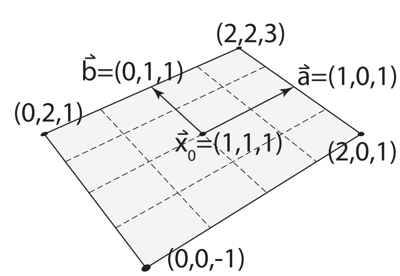
a) Show that if $\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{0}$ (ie. $\boldsymbol{E}$ is *irrotational*), then $\boldsymbol{E} = -\boldsymbol{\nabla} V$ (ie. $\boldsymbol{E}$ is *conservative*), with the potential function $V(\boldsymbol{r}) = -\int_r \boldsymbol{E}\cdot\boldsymbol{dl} + C$, where C is any constant. Show that for a radial path in spherical coordinates, this reduces to $\boldsymbol{V} = -\boldsymbol{r}\cdot\int_0^1 \boldsymbol{E}(\lambda\boldsymbol{r})d\lambda$.

b) Show that if $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$ (ie. $\boldsymbol{B}$ is *incompressible*), then $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$ (ie. $\boldsymbol{B}$ is *solenoidal*), with the potential function $\boldsymbol{A}\cdot\boldsymbol{dl} = \int_r \boldsymbol{B}\cdot\boldsymbol{da} + d\chi$, where the 'constant of integration' $d\chi$ is the differential of any scalar field. Show that for a radial path in spherical coordinates, this reduces to $\boldsymbol{A} = -\boldsymbol{r}\times\int_0^1 \lambda B(\lambda\boldsymbol{r})d\lambda$.

c) [bonus: Show that since the differential of any density field $\rho(\boldsymbol{r})d\tau$ is zero, it can be written as the exact differential $\rho = \boldsymbol{\nabla}\cdot(\boldsymbol{B} + \nabla\times\boldsymbol{A})$ of some flux field $\boldsymbol{B}(\boldsymbol{r})\cdot\boldsymbol{da}$, and any constant of integration (gauge) $\boldsymbol{A}(\boldsymbol{r})\cdot\boldsymbol{dl}$. Derive the formula for $\boldsymbol{B}$ for a radial path in spherical coordinates.]

3. Stokes' theorems–the FTD also generalizes the FTC, integrating the derivative of a field over a region out to the boundary, resulting in an integral of the original field on the boundary.

a) Integrate ${}_{\mathcal{S}}\int \boldsymbol{\nabla}\times\boldsymbol{v}\cdot\boldsymbol{da}$ where $\boldsymbol{v} = \hat{\boldsymbol{x}}\,x^2 + \hat{\boldsymbol{y}}\,2yz + \hat{\boldsymbol{z}}\,xy$, and $\mathcal{S}$ is the parallelogram in the figure to the right. Integrate ${}_{\partial\mathcal{S}}\oint \boldsymbol{v}\cdot\boldsymbol{dl}$ along the boundary ∂S of $\mathcal{S}$ to verify Stokes' theorem.



b) Verify Stokes' theorem with the function in #1d) on the disk $\rho < R, z = 0$ in the xy-plane centered at the origin.

c) Verify Gauss' theorem with the function in #1e) inside the sphere $r < R$ of radius R centered at the origin.