University of Kentucky, Physics 306
Homework #2, Rev. A, due Wednesday, 2024-01-24

1. Complex algebra

$e^{i\phi} = \cos\phi + i\sin\phi$ leads to the beautiful identity $e^{i\pi} + 1 = 0$ involving the three basic operators and five fundamental constants exactly once each. The purpose of this exercise is to explore implications of this relationship between exponential and trigonometric functions and it's role in rotations.

a) Use the power series of e^x, $\sin\phi$, $\cos\phi$ and the property $i^2 = -1$ to prove *Euler's formula*.

b) Show if $z = x + iy$ where (x, y) are the (real, imaginary) Cartesian coordinates of z in the *complex plane*, then $z = \rho e^{i\phi}$, where (ρ, ϕ) are the (radius, azimuth) polar coordinates of z.

c) The *complex conjugate* z^* is formed by replacing i with $-i$ everywhere in z. The *modulus* $|z| \equiv \sqrt{z^* z}$ is the complex analog of absolute value. Use $z = x + iy = \rho e^{i\phi}$ to show the relations $|z|^2 = z^* z = z z^* = x^2 + y^2 = \rho^2$. Thus $|z| = \rho$ is the distance of z from the origin (radius).

d) Show that iz rotates z by 90° CCW about the origin. Use the *arc length formula* $ds = \rho d\phi$ to show that the operator $(1 + id\phi)$ multiplied by z rotates it by an angle $d\phi$. Formally integrate the equation $dz = iz d\phi$ to show that the operator $e^{i\phi}$ rotates z by the angle ϕ. Use this result to justify the identity $e^x = \lim_{n\to\infty}(1 + \frac{x}{n})^n$.

e) Expand z^2 in terms of x,y and also ρ,ϕ to see why $|z|^2$ is generally more useful than z^2.

f) Show that the real and imaginary parts of $z_1^* z_2 = z_2 z_1^* = (x_1 + iy_1)^*(x_2 + iy_2)$ match the dot and cross (z-component) products of the vectors $(x_1, y_1)^T$ and $(x_2, y_2)^T$.

g) Multiply $e^{i\phi}$ by its complex conjugate and expand using Euler's formula to prove the relation $\sin^2\phi + \cos^2\phi = 1$. This shows that $e^{i\phi}$ traces out the unit circle in the complex plane.

h) Use Euler's formula on $e^{\pm i\phi}$ to express $\cos\phi$, $\sin\phi$ and $\tan\phi$ in terms of $e^{i\phi}$ and $e^{-i\phi}$.

i) Using the similar definition $\cosh\alpha \equiv \frac{1}{2}(e^\alpha + e^{-\alpha})$ and $\sinh\alpha \equiv \frac{1}{2}(e^\alpha - e^{-\alpha})$, derive the analog of Euler's formula for the *hyperbolic functions*. *Hint: i becomes $\pm$.*

j) Multiply and expand e^α and $e^{-\alpha}$ to derive a similar formula as in part g) for $\cosh\alpha$ and $\sinh\alpha$. This shows that $(\cosh\alpha, \sinh\alpha)$ traces out a hyperbola in the plane.

k) Derive addition formulas for $\cos(\alpha \pm \beta)$ and $\sin(\alpha \pm \beta)$ by multiplying and expanding $e^{i\alpha} \cdot e^{\pm i\beta}$ and then separating the real and imaginary parts. Do the same for the hyperbolic functions.

l) Use $e^{im\phi}$ to obtain *de Moivre's formula* for $\cos(m\phi) + i\sin(m\phi)$ and $\cosh(m\alpha) \pm \sinh(m\alpha)$. Use this formula to expand $\cos 2\phi$ and $\sin 2\phi$.

m) Obtain the derivatives of $\sin\phi$, $\cos\phi$, and $\sinh\alpha$, $\cosh\alpha$ from the derivative of e^x.

2. Vectors

a) In Mathematica, given the vectors $\vec{\boldsymbol{a}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\vec{\boldsymbol{b}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, draw all integer scalar multiples $\vec{\boldsymbol{a}}\alpha$ and $\vec{\boldsymbol{b}}\beta$ as arrows starting from the origin, illustrate all sums by joining each vector head-to-tail. Plot all linear combinations $\vec{\boldsymbol{a}}\alpha + \vec{\boldsymbol{b}}\beta$ as points (position vectors) and label each point by its components $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ in the basis $(\vec{\boldsymbol{a}} \quad \vec{\boldsymbol{b}})$. Find the components of the vector $\vec{\boldsymbol{v}} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ in the basis $(\vec{\boldsymbol{a}} \quad \vec{\boldsymbol{b}})$ by plotting it and looking up the components on the grid.

Here are some code snipts to help:

```
Graphics[{PointSize[Large],Point[{0,0}],Text[label,{.1,-.1}],Arrow[{{0,0},{2,1}}]},
Axes->True,AxesLabel->{Style[x,Large],Style[y,Large]}]
```

```
Table[{i,j},{i,-1,1},{j,3,5}]
```

`Join[{1},{2,3}]` or `{1,2}~Join~{3}` or `{1,2}<esc>un<esc>{3}`

b) Find the components of the vector $\vec{\boldsymbol{r}} = \begin{pmatrix} x \\ y \end{pmatrix}$ in the basis $(\vec{\boldsymbol{a}} \quad \vec{\boldsymbol{b}})$ by solving the system of two equations $\vec{\boldsymbol{r}} = \vec{\boldsymbol{a}}\alpha + \vec{\boldsymbol{b}}\beta$ (one for x and one for y) for α and β. Now solve the same equation written in matrix form $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ using Mathematica, using `Inverse[{{1,2},{2,1}}]\\MatrixForm` to find the inverse and `{{a,b},{c,d}}.{e,f}}` to multiply matrices.

c) In the vector space of fruit ($\vec{\boldsymbol{a}}$ is one apple, $\vec{\boldsymbol{b}}$ is one banana, $\vec{\boldsymbol{c}}$ is one cherry), a company markets three specialty fruit-baskets: the health-basket with three apples and a banana, ie. $\vec{f}_1 = \vec{a}\,3 + \vec{b}\,1 + \vec{c}\,0$, the sports-basket with an apple, two bananas and a cherry, ie. $\vec{f}_2 = \vec{a}\,1 + \vec{b}\,2 + \vec{c}\,1$, and the red-basket with just an apple and three cherries, ie. $\vec{f}_3 = \vec{a}\,1 + \vec{b}\,0 + \vec{c}\,3$. These baskets can be represented by the basis tranformation matrix $(\vec{f}_1 \quad \vec{f}_2 \quad \vec{f}_3) = (\vec{a} \quad \vec{b} \quad \vec{c}) \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$.

If their total inventory is 10 apples, 8 bananas, 6 cherries, ie. $\vec{x} = (\vec{a} \quad \vec{b} \quad \vec{c}) \begin{pmatrix} 10 \\ 8 \\ 6 \end{pmatrix}$, and they sort it into specialty baskets without wasting any fruit, how many sports-baskets do they need to assemble?