University of Kentucky, Physics 306
Homework #3, Rev. A, due Wednesday, 2024-01-30

1. Orthogonal Projections: Parallel and perpendicular projections can be expressed as $P_\parallel = \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}\cdot$ and $P_\perp = -\hat{\boldsymbol{n}} \times \hat{\boldsymbol{n}}\times$, in terms of the dot and cross products respectively, which themselves are projected products $\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{v}} = av_\parallel$ and $\vec{\boldsymbol{a}} \times \vec{\boldsymbol{v}} = \hat{\boldsymbol{n}}av_\perp$. We will build them up geometrically piece-by-piece, and explore their matrix versions in this problem.

a) Starting with the vector $\vec{\boldsymbol{r}} = \begin{pmatrix} x \\ y \end{pmatrix}$, i) draw $\vec{\boldsymbol{r}}$ for some value of x and y, ii) illustrate the length $r_x = \hat{\boldsymbol{x}} \cdot \vec{\boldsymbol{r}}$, and the vector $\vec{\boldsymbol{r}}_x = \hat{\boldsymbol{x}}\hat{\boldsymbol{x}} \cdot \vec{\boldsymbol{r}}$ with the same magnitude, in the direction $\hat{\boldsymbol{x}}$. iii) Calculate $\vec{\boldsymbol{r}}_x = \hat{\boldsymbol{x}}(\hat{\boldsymbol{x}}\cdot\vec{\boldsymbol{r}})$, where $\hat{\boldsymbol{x}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\hat{\boldsymbol{x}}\cdot = \hat{\boldsymbol{x}}^T = \begin{pmatrix} 1 & 0 \end{pmatrix}$. iv) Calculate the matrix $P_x = \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\cdot$ and then $\vec{\boldsymbol{r}}_x = P_x\vec{\boldsymbol{r}}$, which should be the same by associativity. v) Show that $P_x^2 = P_x$ algebraically and by multiplying matrices. vi) Calculate $P_y = \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}\cdot$ and show that $P_x + P_y = I$, so that $\vec{\boldsymbol{r}}_x + \vec{\boldsymbol{r}}_y = \vec{\boldsymbol{r}}$.

b) Starting with some vector $\vec{\boldsymbol{r}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, i) draw the vectors $\vec{\boldsymbol{r}}$, $\hat{\boldsymbol{n}}r_\perp = \hat{\boldsymbol{z}} \times \vec{\boldsymbol{r}}$ (note both the projection and the rotation by 90°), $-\vec{\boldsymbol{r}}_\perp = \hat{\boldsymbol{z}} \times (\hat{\boldsymbol{z}} \times \vec{\boldsymbol{r}})$ (another rotation) and finally the projection $\vec{\boldsymbol{r}}_\perp$ in the xy-plane, perpendicular to $\hat{\boldsymbol{z}}$. ii) Calculate the components of $\vec{\boldsymbol{r}}_\perp = -\hat{\boldsymbol{z}} \times (\hat{\boldsymbol{z}} \times \vec{\boldsymbol{r}})$ iii) to obtain the matrix $P_\perp = -\hat{\boldsymbol{z}} \times \hat{\boldsymbol{z}}\times$ by its action on $\vec{\boldsymbol{r}}$. iv) Show $P_\perp^2 = P_\perp$ by matrix multiplication and v) $P_\parallel + P_\perp = I$ by adding the matrices, and by using the rule $\vec{\boldsymbol{a}} \times (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) = \vec{\boldsymbol{b}}(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{c}}) - \vec{\boldsymbol{c}}(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}})$.

2. General Projections: relational versus parametric linear/planar geometry.

a) Show graphically that the following equations define the set of points $\{\boldsymbol{x}\}$ on a **line** or **plane**,

	relational	parametric
line	$\{\vec{\boldsymbol{x}} \mid \vec{\boldsymbol{a}} \times \vec{\boldsymbol{x}} = \vec{\boldsymbol{d}}\}$	$\{\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}_1 + \vec{\boldsymbol{a}}\alpha \mid \alpha \in \mathbb{R}\}$
plane	$\{\vec{\boldsymbol{x}} \mid \vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{x}} = D\}$	$\{\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}_2 + \vec{\boldsymbol{b}}\beta + \vec{\boldsymbol{c}}\gamma \mid \beta, \gamma \in \mathbb{R}\}$

where $\vec{\boldsymbol{a}}, \vec{\boldsymbol{d}}, \vec{\boldsymbol{A}}, D$ are constants which define the geometry, $\vec{\boldsymbol{x}}_{1,2}$ are fixed points on the line and plane respectively, and α, β, γ are parameters that vary along the line/plane to uniquely parametrize points in the line/plane. [bonus: Show the 5th relation $\{\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}_2 + \vec{\boldsymbol{A}} \times \vec{\boldsymbol{\delta}} \mid \vec{\boldsymbol{\delta}} \in \mathbb{R}^3\}$.]

b) What constraint between $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{d}}$ is implicit in the formula $\vec{\boldsymbol{a}} \times \vec{\boldsymbol{x}} = \vec{\boldsymbol{d}}$? What is the relation between $\vec{\boldsymbol{b}}$, $\vec{\boldsymbol{c}}$, and $\vec{\boldsymbol{A}}$? Substitute $\vec{\boldsymbol{x}}$ of each parameterization into its relational equation to show the consistency between the two forms and to derive $\vec{\boldsymbol{d}}$ and D in terms of $\vec{\boldsymbol{a}}, \vec{\boldsymbol{x}}_1$ and $\vec{\boldsymbol{A}}, \vec{\boldsymbol{x}}_2$.

c) Define $\tilde{\boldsymbol{a}} \equiv \vec{\boldsymbol{A}}/(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{A}})$, which is parallel to $\vec{\boldsymbol{A}}$, and *normalized* in the sense that $\vec{\boldsymbol{a}} \cdot \tilde{\boldsymbol{a}} = 1$. Using the BAC-CAB rule, show that $\vec{\boldsymbol{x}} = \vec{\boldsymbol{a}}(\tilde{\boldsymbol{a}} \cdot \vec{\boldsymbol{x}}) - \tilde{\boldsymbol{a}} \times (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{x}})$ for all $\vec{\boldsymbol{x}}$. Illustrate this non-orthogonal projection of $\vec{\boldsymbol{x}}$ onto vectors $\vec{\boldsymbol{x}}_1$ parallel to the line plus $\vec{\boldsymbol{x}}_2$ parallel to the plane. [bonus: use this to calculate the point $\vec{\boldsymbol{x}}_0$ at the intersection of the line and plane in terms of $\vec{\boldsymbol{a}}, \vec{\boldsymbol{d}}, \vec{\boldsymbol{A}}, D$. Verify this by showing that $\vec{\boldsymbol{x}}_0$ satisfies the relational equation for both the line and plane.]

d) Let $\tilde{\boldsymbol{a}} \equiv \frac{\vec{\boldsymbol{b}}\times\vec{\boldsymbol{c}}}{\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}}\times\vec{\boldsymbol{c}}}$, $\tilde{\boldsymbol{b}} \equiv \frac{\vec{\boldsymbol{c}}\times\vec{\boldsymbol{a}}}{\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}}\times\vec{\boldsymbol{c}}}$, and $\tilde{\boldsymbol{c}} \equiv \frac{\vec{\boldsymbol{a}}\times\vec{\boldsymbol{b}}}{\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}}\times\vec{\boldsymbol{c}}}$, where the arrows on $\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}}$ distinguish them from their *covectors* $\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}, \tilde{\boldsymbol{c}}$. The definition of $\tilde{\boldsymbol{a}}$ is the same as in c) with $\vec{\boldsymbol{A}} = \vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}$. Calculate the nine combinations of $(\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}})^T \cdot (\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}, \tilde{\boldsymbol{c}}) = I$, i.e. $\vec{\boldsymbol{a}} \cdot \tilde{\boldsymbol{a}} = 1$, $\vec{\boldsymbol{a}} \cdot \tilde{\boldsymbol{b}} = 0$, ... to show they are *mutually orthonormal*. In this sense, $(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}, \tilde{\boldsymbol{c}})$ is the *dual* basis of $(\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}})$. [bonus: Show using Cramer's rule that $[\tilde{\boldsymbol{a}}|\tilde{\boldsymbol{b}}|\tilde{\boldsymbol{c}}] = [\vec{\boldsymbol{a}}|\vec{\boldsymbol{b}}|\vec{\boldsymbol{c}}]^{-1}$, ie. that it is a *reciprocal basis*.]

e) The *contravariant components* of $\vec{\boldsymbol{x}}$ are defined as the components (α, β, γ) that satisfy the equation $\vec{\boldsymbol{x}} = \vec{\boldsymbol{a}}\alpha + \vec{\boldsymbol{b}}\beta + \vec{\boldsymbol{c}}\gamma$; i.e., $(\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}})$ is the contravariant basis. Using $\vec{\boldsymbol{x}} \cdot \tilde{\boldsymbol{a}}$, etc., calculate the three contravariant components of $\vec{\boldsymbol{x}}$ in terms of dot products. Likewise, find the *covariant components* $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ of $\vec{\boldsymbol{x}}$, defined as the components $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ which satisfy $\vec{\boldsymbol{x}} = \tilde{\boldsymbol{a}}\tilde{\alpha} + \tilde{\boldsymbol{b}}\tilde{\beta} + \tilde{\boldsymbol{c}}\tilde{\gamma}$; i.e. $(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}, \tilde{\boldsymbol{c}})$ is the covariant basis.