University of Kentucky, Physics 306
Homework #5, Rev. B, due Wednesday, 2024-02-14

1. Stretches. In analogy with the *polar decomposition* $w = x + iy = \rho e^{i\phi}$ of complex numbers, any matrix A can be decomposed $A = RS$ into a stretch S and a rotation R, which are the building blocks of all linear operators. This problem set explores the structure of stretches.

a) For any eigenvalue problem $M\vec{\boldsymbol{v}}_i = \vec{\boldsymbol{v}}_i\lambda_i$, augment these n equations to obtain $MV = VW$, where $W = \text{diag}(\lambda_1, \lambda_2, \ldots)$, and thus show $M = VWV^{-1}$ and $W = V^{-1}MV$. This is the *similarity transform* to the eigenbasis of M, in which the operator's matrix is diagonal, and its inverse.

b) Diagonalize σ_x and σ_y by calculating their eigenbasis $U_{x,y} = (\vec{v}_1|\vec{v}_2)$ of H04#1c to justify the similarity transforms $\sigma_x U_x = U_x \sigma_z$ of H04#1d. Thus, $\sigma_{x,y}$ are matrices of the same operator σ_z in different bases. Calculate the eigenvalues and eigenvectors of σ_z, which is already diagonal.

c) Calculate the eigenvalues and eigenvectors of the *projections* $P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ of H04#1b. [*bonus*: show that projections, $P^2 = P$, can only have eigenvalues of 0 or 1.]

d) Calculate all eigenvalues and eigenvectors of the *ladder operators* $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$ of H04#1b to show that they are not diagonalizable. Such matrices with less eigevectors than eigenvalues are called *defective* (see H08#2). Show that they are also *nilpotent*: $\sigma_\pm^n = 0$ for some value n.

e) [*bonus*: A *symmetric* matrix S is guaranteed to have a complete set of eigenvectors $V = (\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \ldots)$ and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots$, such that $S\vec{\boldsymbol{v}}_i = \vec{\boldsymbol{v}}_i\lambda_i$ with the following special properties: show that i) the eigenvalues of S must be real: $\lambda^* = \lambda$, and that ii) two eigenvectors $\vec{\boldsymbol{v}}_i, \vec{\boldsymbol{v}}_j$ of S with distinct eigenvalues $\lambda_i \neq \lambda_j$ must be orthogonal: $\vec{\boldsymbol{v}}_i \cdot \vec{\boldsymbol{v}}_j = 0$. Thus the matrix of eigenvectors is unitary: $V^\dagger V = I$, so that $S = VWV^\dagger$ and $W = V^\dagger SV$. Interpret these similarity transforms geometrically.]

f) In addition to symmetric matrices, complex *normal matrics*, $N^\dagger N = NN^\dagger$ (see H08#1), also have an othogonal eigenbasis but only have real eigenvalues if they are *Hermitian*, $H^\dagger = H$. Calculate the eigenvalues and eigenvectors of $M_z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ Do they look familiar? Show that $M_z = VWV^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ and $e^{M_z\phi} = Ve^{W\phi}V^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$. Multiply this out to verify H04#2b. This is an example of the *normal matrix analogy*, which relates matrices and complex numbers. Hermitian matrices have real eigenvalues while anti-Hermitian matrices have zeros on the diagonal and imaginary eigenvalues. The exponential of a Hermitian matrix is positive definite and has real positive eigenvalues, while the exponential of an anti-Hermitian matrix is unitary with unit modulus eigenvalues and therefore determinant.

2. [*bonus*: We saw in H04#1 that the *trace* and *determinant* are matrix **invariants** under similarity transformation. In general, an $n \times n$ matrix has n independent invariants, including the trace and determinant.

a) Show that the *characteristic equation* $|A - \lambda I| = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$, substituting $\lambda \to A$ in the second equality, is invariant under similarity transforms. Thus A has n independent invariants: either $a_0 \ldots a_{n-1}$ of this equation or its n eigenvalue roots. The first and last coefficients a_{n-1} and a_0 are $\text{tr}(A)$ and $\det(A)$, respectively, while the others are k-dimensional 'perimeters', for example, *surface area*, of the transformation.

b) Prove the Cayley-Hamilton Theorem, that any matrix A satisfies its own characteristic equation, for the case of diagonal matrices. The full theorem follows from the invariance of the characteristic equation. Show that $A^{-1} = -(A^{n-1} + a_{n-1}A^{n-2} + \ldots + a_1 I)/a_0$.]