University of Kentucky, Physics 306
Homework #6, Rev. A, due Wednesday, 2024-02-21

1. Finite dimensional function spaces.

a) In the space of cubic polynomials with basis vectors $f_0(x) = 1$, $f_1(x) = x$, $f_2(x) = x^2$, $f_3(x) = x^3$, i) what are the components of the function $f(x) = a + bx + cx^2 + dx^3$? ii) Find the components of the same function in the basis $P_0 = 1$, $P_1 = x$, $P_2 = \frac{1}{2}(3x^2 - 1)$, $P_3 = \frac{1}{2}(5x^3 - 3x)$.

b) Calculate the matrix D of the derivative operator d/dx, in i) the basis $[f_0, f_1, f_2, f_3]$, and ii) in the basis $[P_0, P_1, P_2, P_3]$. iii. [*bonus*: Show that D is nilpotent.]

c) Find the components of the function $A\cos(x - \phi)$ i) in the basis $h_1 = \cos(x)$, $h_2 = \sin(x)$, and ii) in the basis $k_1 = e^{ix}$, $k_2 = e^{-ix}$.

2. The **Fourier Series** is an application of using an orthogonal basis to extract components of a generalized vector (function). Orthogonality is crucial, because otherwise it would be impossible to solve the linear system of equations (invert an infinite-dimensional matrix) to obtain each component.

a) Submit a screenshot of your completed Level 3 in the Fourier: Making Waves Wave Game.

b) Calculate the inner products $\langle n|m\rangle = \int_{-\pi}^{\pi} d\phi\, \Phi_n^*(\phi)\Phi_m(\phi)$ of the Fourier sine basis functions $\Phi_n(\phi) = \frac{1}{\sqrt{\pi}}\sin(n\phi)$. Use this orthonormal basis to calculate the components b_n of the Fourier sine series $f(\phi) = \sum_{n=1}^{\infty} b_n\Phi_n(\phi)$ of the function $f(\phi) = 1$ for $0 < \phi < \pi$ and $f(\phi) = -1$ for $-\pi < \phi < 0$. Plot the first $1, 2, 3, 5, 10$ terms in Mathematica. What does the periodic series look like outside the interval $-\pi < \phi < \pi$? Compare the inner product $\langle f|f\rangle = \int_{-\pi}^{\pi} d\phi\, f^*(\phi)f(\phi)$ of $f(\phi)$ with that of its Fourier components $\langle f|f\rangle = \sum_n b_n^* b_n$. [it's a **rotation** from the $|\phi\rangle$ to $|n\rangle$ basis!]