University of Kentucky, Physics 306
Homework #8, Rev. A, Bonus

1. The good, ...

a) The *commutator* $[A, B]$ of two matrices A and B is defined as the matrix $[A, B] \equiv AB - BA$. Calculate $[\sigma_j, \sigma_k]$ for the *Pauli matrices* σ_i, $i = x, y, z$, using H04#1a. Do they commute?

b) Show that the commutator of A and B is antisymmetric: $[A, B] = -[B, A]$, and bilinear: $[\alpha_i A_i, B_j \beta_j] = \alpha_i [A_i, B_j] \beta_j$. Thus it is a 'matrix version' of the vector cross product: $[A, A] = 0$ and two matrices commute if they are in the 'same direction' (both diagonal in the same basis).

c) Verify that the two matrices $A = \begin{pmatrix} 9 & -2 & -6 \\ -2 & 9 & -6 \\ -6 & -6 & -7 \end{pmatrix}$ and $B = \begin{pmatrix} 54 & 10 & -3 \\ 10 & -45 & 30 \\ -3 & 30 & 46 \end{pmatrix}$.
commute. Simultaneously diagonalize both matrices. Are the eigenvectors orthogonal?
Hint: Octave or Mathematica is your friend!

d) By adding and subtracting $M = H + iK$ and its adjoint $M^\dagger = H - iK$, where $H^\dagger = H$ and $K^\dagger = K$ are Hermitian (note that $A = iK$ is antisymmetric: $A^\dagger = -A$), show that any matrix M can be decomposed into its symmetric and antisymmetric parts, similar to how $z = x + iy$ projects into its real and imaginary parts.

e) A *normal* matrix is one that commutes with its adjoint: $[N^\dagger, N] = 0$. Setting $N = H + iK$ as in d), show that $[N^\dagger, N] = 2i[H, K]$. This shows that a normal matrix is one whose symmetric and antisymmetric parts commute. Thus normal matrices are the most general matrices that can be diagonalized with orthogonal eigenvectors. The eigenvalues of N are $n_j = h_j + ik_j$ where h_j and k_j are the eigenvalues of H and K. Thus N acts like independent complex numbers!

f) What are the properties of the eigenvalues of the following types of matrices: i) [anti]Hermitian, $H^\dagger = \pm H$; ii) Unitary, $U^\dagger = U^{-1}$; iii) Zero, $Z\vec{v} = \vec{0}$; iv) Identity, $I\vec{v} = \vec{v}$; v) Traceless, $\mathrm{tr}\, A = 0$; vi) Special, $\det A = 1$; vii) Projection, $P^2 = P$; viii) Involution, $A = A^{-1}$.

g) In analogy with polar complex numbers $z = e^{\sigma + i\phi} = \rho u$, where $\rho = e^\sigma > 0$ and $u = e^{i\phi}$ is a *unit*: $|u|^2 = u^* u = 1$, any matrix A has a *polar decomposition* $A = RS$ into a rotation R and a stretch S, which are the building blocks of all linear operators. By taking the exponential $M = e^N$ of $N = H + iK$, show that any normal matrix can be decomposed into $M = UP$, where U is unitary, $U^\dagger U = I$, and P is positive definite, $P^\dagger = P$ with non-negative eigenvalues.

h) Using the polar decomposition $A = RS$, where $R^\dagger R = I$ and $S^\dagger = S$, and diagonalizing $S = VWV^\dagger$, show that any matrix may be decomposed into $A = UWV^\dagger$, where $U = RV$ and V are unitary, and W is diagonal. This *singular value decomposition* of matrices is the generalization the spectral theorem for all matrices. Because it involves both stretches and rotations, every *singular value* (eigenvalue) w_i has two eigenvectors: a *left singular vector* $\vec{u}_i$ and a *right singular vector* $\vec{v}_i$. This is useful for transformations from one vector space into another, not just operators.

2. the bad, . . .

a) The Pauli *ladder operators* $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$ transform the unit vectors $(1,0)^T$ and $(0,1)^T$ from one to the next along a chain, *annihilating* the last one. Calculate $\sigma_\pm$, $\sigma_\pm^2$, $\sigma_\pm^3, \ldots, \sigma^n$. This is a *nilpotent* matrix: $Z^n = 0$ for some n. By definition, $Z^0 = I$ is the identity.

b) Calculate the doubly *degenerate* eigenvalues λ of $\sigma_\pm$ and show that they have only one eigenvector $\vec{v}_1$. A matrix with less eigenvectors than eigenvalues is called *defective*, and is the unique sum of two commuting matrices: one *diagonalizable* and the other *nilpotent* by the *Jordan-Chevalley* decomposition theorem. The other vectors associated with λ are called *generalized eigenvectors*, defined by $M\vec{v}_k = \vec{v}_k\lambda + \vec{v}_{k-1}$. Identify each term with the corresponding matrix part above. Note that $M\vec{v}_1$ follows the pattern if we define $\vec{v}_0 = \vec{0}$. Find the matrix J (called the *Jordan block*) of M in the basis $V = \{\vec{v}_k\}$, used in the *Jordan normal form* (similarity transform) $M = VJV^{-1}$.

c) By expanding the Taylor series, show that $f(A) = Vf(J)V^{-1} = V\left(\bigoplus_{k=1}^N f\left(J_{\lambda_k,m_k}\right)\right)V^{-1}$, where $A = VJV^{-1} = V\left(\bigoplus_{k=1}^N J_{\lambda_k,m_k}\right)V^{-1}$ and

$$f(J_{\lambda,n}) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\lambda)Z^k}{k!} = \begin{pmatrix} f(\lambda) & f'(\lambda) & \frac{f''(\lambda)}{2} & \cdots & \frac{f^{(n-2)}(\lambda)}{(n-2)!} & \frac{f^{(n-1)}(\lambda)}{(n-1)!} \\ 0 & f(\lambda) & f'(\lambda) & \cdots & \frac{f^{(n-3)}(\lambda)}{(n-3)!} & \frac{f^{(n-2)}(\lambda)}{(n-2)!} \\ 0 & 0 & f(\lambda) & \cdots & \frac{f^{(n-4)}(\lambda)}{(n-4)!} & \frac{f^{(n-3)}(\lambda)}{(n-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & f(\lambda) & f'(\lambda) \\ 0 & 0 & 0 & \cdots & 0 & f(\lambda) \end{pmatrix}$$

[wiki: Jordan Matrix].

d) Calculate the adjoint $\sigma_\pm^\dagger$, the *projections* $P_+ = \sigma_+\sigma_-$ and $P_- = \sigma_-\sigma_+$, and the commutators $[\sigma_+, \sigma_-]$, $[I, \sigma_\pm]$, and $[\sigma_z, \sigma_\pm]$. Note that $\sigma_\pm$ commutes with I, which has all the same eigenvalues, but not with σ_z, which has distinct eigenvalues. This is true in general. Use the last commutation relation to show that if $\vec{v}$ is an eigenvector of σ_z, then $\sigma_\pm\vec{v}$ is also an eigenvector. Thus $\sigma_\pm$ raises and lowers one one eigenvector of σ_z into another, like climbing up or down the rungs of a ladder, which explains their names.

e) Show that $\sigma_x\sigma_\pm\sigma_x^{-1} = \sigma_\mp$ Find the most general 2×2 nilpotent operator via the similarity transform $V\sigma_+V^{-1}$ of σ_+ into a generic basis $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Represent it as an outer product. What is its geometrical interpretation? Do the same for a generic *diagonalizable* operator to represent it as the sum of two outer products (its *spectral decomposition*).

f) The derivative operator $D = d/dx$ is a nilpotent operator in the linear space of polynomials $A(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$. i) Find the matrix of D, in the basis of monomials $\hat{\boldsymbol{e}}_k = x^k$. Show that $D^{n+1} = 0$. ii) What is the eigenvalue of D in this space? iii) In what basis is its matrix in the normal Jordan form described in b)? Show how this relates to the Taylor series $A(x) = \sum_k \frac{1}{k!}A^{(k)}(0)x^k$. iv) D acts like the identity on e^x and thus is not nilpotent in this space. What is different about e^x? v) What are the eigenvalues and eigenfunctions of D?

3. . . . and the gnarly.

We have seen that diagonal matrices behave nicely, adding *and* multiplying like n independent complex numbers. By extension, a function of any diagonalizable matrix acts independently on each eigenvalue in its eigenbasis. We saw in #1 that commuting matrices can be *simultaneously diagonalized* to perform operations in their common diagonal basis. This leads to the *normal matrix analogy*, where normal matrices act like n independent complex numbers in the common orthogonal eigenbasis of their *Hermitian* plus *anti-Hermitian*, or *positive semidefinite* times *unitary* parts.

On the other hand we saw in #2 that matrices with a *nilpotent* component were particularly nasty, and could not be diagonalized at all, but only reduced to the *Jordan normal form.* These *defective matrices* are worse than *singular matrices*, which are still potentially diagonalizable, but cannot be inverted because of one or more null eigenvalues $\lambda_k = 0$.

Only a small slice of matrices are 'bad' (defective), while an even smaller sliver are 'good' (normal). But what about the vast expanse of 'gnarly' (abnormal but diagonalizable) matrices, which are neither 'good' nor 'bad'? In this exercise, we will show there exists a *generalized adjoint*, in which any such matrix is normal, with an orthonormal eigenbasis in the corresponding metric (the *generalized eigenvalue problem*). Thus, *"beauty is in the eye of the beholder"*, and any non-defective matrix is normal in its natural setting of appropriately defined distance and angle.

And what about pairs of matrices that do not commute? For these we consider a second form of *simultaneous diagonalization*, which is possible for any two Hermitian matrices. We diagonalize one matrix, treating is as a *metric*, with a *congruency transform* $I = V^\dagger g V$, resulting in the *identity* matrix I, signifying an orthonormal basis. This is always preserved during diagonalization of the second *operator* matrix using the more common *similarity transform* $D = V^{-1} M V$. Note that eigenvalues of g can always be normalize to 1 by scaling the eigenvectors V, while eigenvalues of M are invariant because $V^{-1}V = I$ always by definition.

a) Given an eigenbasis V of a matrix $M = VDV^\dagger$, describe a procedure for finding the class of metrics g in which V is orthonormal: $V \cdot V = V^\dagger g V = I$. This can be thought of as the inverse of the *Gram-Schmidt* method. Show that M is normal using the *adjoint* $M^\dagger \equiv g^{-1} M^T g$. For what other metrics is M normal? Show that if $M\vec{\boldsymbol{v}}_i = g\vec{\boldsymbol{v}}_i\lambda_i$ or $MV = gVD$ (the *generalized eigenvalue problem*), then $V^\dagger g V = I$, so V can be chosen to be orthonormal with respect to the metric g.

b) Working backwards, show that any two Hermitian matrices $A^\dagger = A$ and $B^\dagger = B$ can be simultaneously diagonalized in the following form: $V^\dagger A V = D$ and $V^\dagger B V = I$. This is useful to solve a system of particles at coordinates x^k, connected with masses and springs, having kinetic plus potential energy $H = \frac{1}{2}\dot{\vec{\boldsymbol{x}}}^\dagger M \dot{\vec{\boldsymbol{x}}} + \frac{1}{2}\vec{\boldsymbol{x}}^\dagger K \vec{\boldsymbol{x}}$. Here, M acts as the metric for $T = \frac{1}{2}mv^2 = \frac{1}{2}\dot{\boldsymbol{x}} \cdot \dot{\boldsymbol{x}}$.