University of Kentucky, Physics 306
Homework #9, Rev. A, due Wednesday, 2024-03-20

1. Solve Laplace's equation $\nabla^2 V = (\partial_x^2 + \partial_y^2)V = 0$ for the **tent potential** $V(x, y)$ defined on the region $-a<x<a$ and $-b<y<b$ with boundary conditions $V(x, \pm b) = 0$ and $V(\pm a, y) = V_0(1-|y/b|)$ by following these steps:

a) Substitute the eigenfunctions $X(x)$, $Y(x)$ and eigenvalues $-k_x^2$, $-k_y^2$ of the two differential operators ∂_x^2, ∂_y^2 of Laplace's equation to get the *disperson relation* between the two eigenvalues.

b) Apply boundary conditions at $y = \pm b$ to quantize k_n, $Y_n(y)$ and therefore $X_n(x)$, and form a linear combination of all eigenfunctions to obtain the general solution $V(x, y) = \sum_n c_n X_n(x) Y_n(y)$.

c) Apply boundary conditions at $x = \pm a$ to solve the coefficients c_n of the general solution.

d) Sketch the solution and its first three Fourier components in Mathematica.

2. Conformal maps [*bonus*: in contrast to vector spaces, which have a linear (parallel) structure, curvilinear coordinate systems (parametrizations of points in space) can have curves and surfaces of any nondegenerate shape (every point has unique coordinates, except possibly a few singularities). A coordinate transformation is a multidimensional function $f : (x, y, z) \to (u, v, w)$ or its inverse $f^{-1} : (u, v, w) \to (x, y, z)$ between two coordinates of the same point. In 2d these can be represented by complex functions $w = u + iv = f(x + iy) = f(z)$.

An orthogonal coordinate system is one in which the coordinate lines and surfaces intersect at right angles. In 2d, this is automatically satisfied by *analytic* functions, which formally depend only on z and not z^*, for example, any combination of algebraic or trigonometric functions of z. In fact, both $u(x, y)$ and $v(x, y)$ satisfy the Laplace equation, so that the contours of these function represent physical potentials and field lines, respectively in 2d. These functions can be composed to create most common coordinate systems in 2d.

For each of the functions $w = f(z)$ mapping $z = x + iy \mapsto w = u + iv$, plot: i) the contours of $u(x, y)$ and $v(x, y)$ in the z-plane; and ii) the two families of curves $f(x + iy)$, parameterized by constant x or y respectively, in the w-plane (the inverse transformation).

a) $f(z) = z + c$ for a complex constant $c = a + ib$ (use $c = 2 + i$).

b) $f(z) = cz$ for the same constant c.

c) $f(z) = z^2$. Show the two branches in the z-plane and the corresponding branch cut in w.

d) $f(z) = e^z$, the transformation to polar coordinates $z = \rho e^{i\phi}$.

e) $f(z) = \cosh(z)$, the transformation to elliptical coordinates.

f) $f(z) = e^z + z$. A contour of this function was used by Rogowski to create a smooth edge of and electrode without any "hot spots" of high electric field which would arc. The other family of contours represents the lines of electric flux ending at charges on the electrode.]

3. Vectors in **curvilinear coordinates** (q^1, q^2, q^3) have a natural *coordinate basis* $\boldsymbol{b}_i \equiv \partial\boldsymbol{r}/\partial q^i$ and *reciprocal basis* $\boldsymbol{b}^i \equiv \boldsymbol{\nabla} q^i = \partial q^i/\partial\boldsymbol{r}$. Each basis vector is a *vector field* (a function of position). The most common coordinate systems are Cartesian $q^i = (x, y, z)$, cylindrical $q^i = (\rho, \phi, z)$, and spherical $q^i = (r, \theta, \phi)$, defined by the transformations $x + iy = \rho e^{i\phi}$ and $z + i\rho = re^{i\theta}$, respectively. These are all orthogonal, right-handed systems, for which both bases are aligned with the *orthonormal basis* $\hat{\mathbf{e}}_i = \boldsymbol{b}_i/h_i = \boldsymbol{b}^i h_i$, where $h_i = |\boldsymbol{b}_i| = 1/|\boldsymbol{b}^i|$ is called the *scale factor*.

a) Determine the coordinate transformation $q^i(q^{i'})$ from each coordinate system to each of the others. *Hint: invert and combine the two transformations above.*

b) For each coordinate system, illustrate the three coordinate isosurfaces $q^i(\boldsymbol{r}) = q_0^i$ (constant) passing through an arbitrary point $\boldsymbol{r}_0$, labeling lengths and angles in your diagram. For each coordinate q^i, identify the curve $\boldsymbol{s}(q^i; q_0^j, q_0^k)$ at the intersection of two surfaces of constant $q^j = q_0^j$ and $q^k = q_0^k$.

c) For each coordinate system, calculate $\boldsymbol{b}_i = \partial\boldsymbol{r}/\partial q^i$ using $\boldsymbol{dr} = \hat{\boldsymbol{x}}dx + \hat{\boldsymbol{y}}dy + \hat{\boldsymbol{z}}dz$. Calculate the *metric* $g_{ij} = \boldsymbol{b}_i \cdot \boldsymbol{b}_j = \mathrm{diag}(h_1^2, h_2^2, h_3^2)$ and normalize $\boldsymbol{b}_i = \hat{\mathbf{e}}_i h_i$ to find the *unit vectors*. The scale factors h_θ and h_ϕ for angular coordinates are just the radii of curvature, according to the arc length formulae $ds_\theta = rd\theta$ and $ds_\phi = \rho d\phi$.

d) Construct the transformation matrices between unit bases, by considering rotations $R_z(\phi)$ (rotation by an angle ϕ about the z-axis) and $R_\phi(\theta)$ (about the y-axis). Compare with part c).

e) For each coordinate system, calculate the *line element* $\boldsymbol{dl} = \hat{\mathbf{e}}_i h_i dq^i$, the *area element* $\boldsymbol{da} = \frac{1}{2}\boldsymbol{dl} \times \boldsymbol{dl} = \hat{\mathbf{e}}_k\, h_i h_j\, dq^i dq^j$, and the *volume element* $d\tau = \frac{1}{3}\boldsymbol{dl} \cdot \boldsymbol{da} = h_1 h_2 h_3\, dq^1 dq^2 dq^3$.

f) Use the coordinate transformations $(\rho, \phi, z) = f^{-1}(x, y, z)$ from a) to calculate the covariant basis $\boldsymbol{b}^i = \nabla q^i = \hat{\mathbf{e}}_i/h_i$ and verify that $\boldsymbol{b}_i \cdot \boldsymbol{b}^j = \delta_i^j$. Calculate $g^{ij} = \boldsymbol{b}^i \cdot \boldsymbol{b}^j = \mathrm{diag}(h_1^{-2}, h_2^{-2}, h_3^{-2})$. [bonus: Do the same for spherical coordinates.]