University of Kentucky, Physics 306
Homework #10, Rev. A, due Wednesday, 2024-03-27

1. The **gradient**, **curl**, and **divergence** are all applications of d in different dimensions. They are defined as $df = \boldsymbol{\nabla} f \cdot \boldsymbol{dl}$, $d(\boldsymbol{A}\cdot\boldsymbol{dl}) = (\boldsymbol{\nabla}\times\boldsymbol{A})\cdot\boldsymbol{da}$, $d(\boldsymbol{B}\cdot\boldsymbol{da}) = (\boldsymbol{\nabla}\cdot\boldsymbol{B})\cdot d\tau$, respectively, in terms of the differential operator $d = dq^i\partial_i$ and elements $\boldsymbol{dl} = \hat{\boldsymbol{e}}_i h_i dq^i$, $\boldsymbol{da} = \frac{1}{2}\boldsymbol{dl}\times\boldsymbol{dl}$, $d\tau = \frac{1}{3}\boldsymbol{dl}\cdot\boldsymbol{da}$.

a) Apply these definitions to scalar f, polar vector $\boldsymbol{A}$, and axial vector $\boldsymbol{B}$ fields to obtain $\boldsymbol{\nabla} f = \frac{df}{\boldsymbol{dr}} = \frac{\hat{\boldsymbol{e}}_i}{h_i}\frac{\partial}{\partial q^i} f$, $\quad \boldsymbol{\nabla}\times\boldsymbol{A} = \frac{d\,\boldsymbol{dr}\cdot}{\boldsymbol{da}}\boldsymbol{A} = \frac{\hat{\boldsymbol{e}}_i}{h_j h_k}\frac{\partial}{\partial q^j} h_k A_k$, $\quad \boldsymbol{\nabla}\cdot\boldsymbol{B} = \frac{d\,\boldsymbol{da}\cdot}{\boldsymbol{d\tau}}\boldsymbol{B} = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial q^k} h_i h_j B_k$, where i, j, k are cyclic and $\frac{1}{\boldsymbol{dr}}$ is the inverse transformation of $\boldsymbol{dr}\cdot$ from $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$ to $\{dx, dy, dz\}$, etc.

b) Expand a) in Cartesian, cylindrical, and spherical coordinates (compare with this table).

c) Calculate the gradient of $f(r,\theta,\phi) = r^\ell P_\ell(\cos(\theta))$, using $\frac{x^2-1}{\ell}P'_\ell(x) = xP_\ell(x) - P_{\ell-1}(x)$,

d) the curl of $\boldsymbol{A}(x,y,z) = (\hat{\boldsymbol{y}}x - \hat{\boldsymbol{x}}y)/(x^2+y^2)^n$ in Cartesian and cylindrical coordinates, and the

e) divergence of $\boldsymbol{B}(x,y,z) = (\hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y + \hat{\boldsymbol{z}}z)/(x^2+y^2+z^2)^n$ in Cartesian and spherical coordinates.

f) Derive the formulas for $d^2 f$ and $d^2\boldsymbol{A}\cdot\boldsymbol{dl}$ in terms of their partial derivatives to show that the trivial 2^{nd} derivatives $\boldsymbol{\nabla}\times\boldsymbol{\nabla} f$ and $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times\boldsymbol{A}$ are special cases of $d^2 = 0$.

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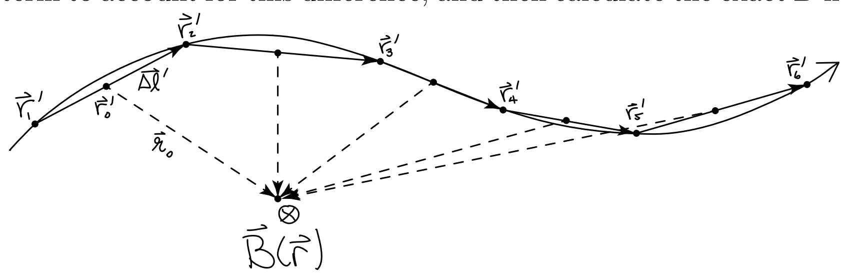
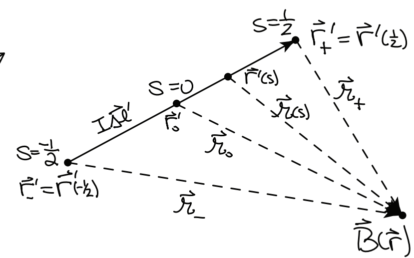
2. The magnetic analog of *Coulomb's law* (with a scalar charge element $dq = \lambda dl = \sigma da = \rho d\tau$) is the **Biot-Savart law** (with a vector current element $\boldsymbol{v}dq = I\boldsymbol{dl} = \boldsymbol{K}da = \boldsymbol{J}d\tau$):

$$\boldsymbol{B} \;=\; \frac{\mu_0}{4\pi}\oint' \frac{\boldsymbol{v}dq' \times \boldsymbol{\mathscr{r}}}{\mathscr{r}^3} \;\approx\; \sum_i \left(\Delta\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{I\Delta\boldsymbol{\ell}\times\boldsymbol{\mathscr{r}}_0}{\mathscr{r}_0^3}\right)_i, \tag{1}$$

where $\Delta\boldsymbol{\ell}$ is the displacement vector from the beginning to the end of each current segment, and $\boldsymbol{\mathscr{r}}_0$ is the displacement vector from the middle of each current segment $\boldsymbol{r}'_0$ to the field point $\boldsymbol{r}$. The approximation is that all of the current is concentrated at $\boldsymbol{r}'_0$ instead of spread out along the length of the segment from $\boldsymbol{r}'_0 - \Delta\boldsymbol{\ell}/2$ to $\boldsymbol{r}'_0 + \Delta\boldsymbol{\ell}/2$. In this problem we first calculate a correction term to account for this difference, and then calculate the exact B-field due to each straight segment.





a) To analytically integrate the Biot-Savart law along a single straight segment of the path, parametrize the segment $\boldsymbol{r}'(s)$ with the parameter s, ranging from $s = -\frac{1}{2}$ at the beginning to $s = +\frac{1}{2}$ at the end of the segment. The parameterization involves the constant vectors $\boldsymbol{r}'_0$ (the center of the segment) and $\Delta\boldsymbol{\ell}$ (displacement along the segment). Calculate the line element $\boldsymbol{dl} = \frac{d\boldsymbol{r}'}{ds}ds$. Calculate $\boldsymbol{\mathscr{r}}$ as a function of $\boldsymbol{\mathscr{r}}_0$, $\Delta\boldsymbol{\ell}$, and s. Substitute these into the Biot-Savart formula and factor out the constant approximation of Eq. 1] to obtain

$$\Delta\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu 0}{4\pi}\frac{I\Delta\boldsymbol{\ell}\times\boldsymbol{\mathscr{r}}_0}{\mathscr{r}_0^3}T(\alpha,\beta), \tag{2}$$

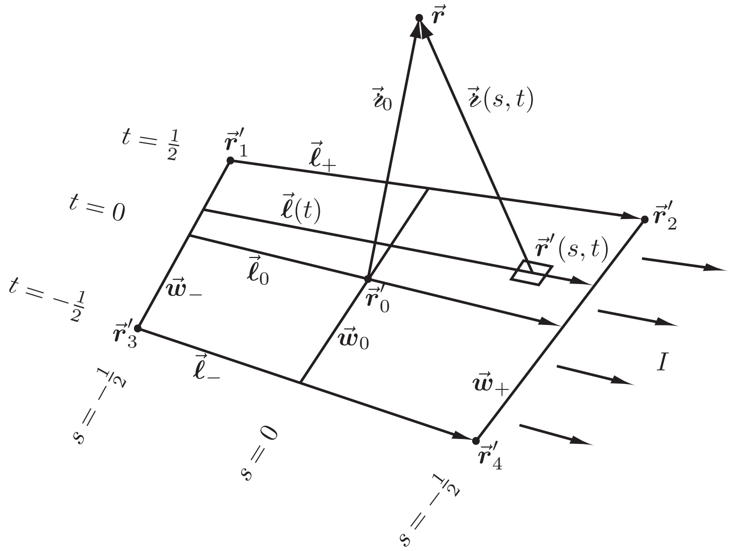
where the integral $T(\alpha,\beta)$ along s depends on $\alpha = \boldsymbol{\mathscr{r}}_0\cdot\Delta\boldsymbol{\ell}/\mathscr{r}_0^2$ and $\beta = \Delta\ell^2/\mathscr{r}_0^2$.

b) [bonus: Approximate the integrand of $T(\alpha,\beta)$ to order s^2 and integrate to obtain the correction term $T(\alpha,\beta) \approx 1 + \frac{1}{8}(5\alpha^2 - \beta)$ for the case where all the current is at the center of the segment.]

c) [bonus: Calculate the exact integral $T(\alpha,\beta)$ and show that $\Delta\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\frac{\Delta\boldsymbol{\ell}\times\boldsymbol{\mathscr{r}}_0}{(\Delta\boldsymbol{\ell}\times\boldsymbol{\mathscr{r}}_0)^2}\,\Delta\boldsymbol{\ell}\cdot(\hat{\boldsymbol{\mathscr{r}}}_- - \hat{\boldsymbol{\mathscr{r}}}_+)$, where $\boldsymbol{\mathscr{r}}_\pm = \boldsymbol{r} - \boldsymbol{r}'(\pm\frac{1}{2})$ is the displacement vector from each end of the segment to the field point and $\hat{\boldsymbol{\mathscr{r}}}_\pm = \boldsymbol{\mathscr{r}}_\pm/\mathscr{r}_\pm$.]

d) [bonus: show that this is equivalent to $\Delta\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\frac{(\boldsymbol{\mathscr{r}}_-\times\boldsymbol{\mathscr{r}}_+)(\mathscr{r}_- + \mathscr{r}_+)}{\mathscr{r}_-\mathscr{r}_+(\mathscr{r}_-\mathscr{r}_+ + \boldsymbol{\mathscr{r}}_-\cdot\boldsymbol{\mathscr{r}}_+)}$.]

[*bonus:* **3. Current sheet**—surface currents can be approximated numerically by a tiling of quadrilaterals like the one shown below, with current I flowing parallel to the top and bottom edges, from left to right. Let the vector $\boldsymbol{\ell} = \boldsymbol{\ell}_+ = \boldsymbol{\ell}_-$ run along either the top or bottom edge, parallel to the current; $\boldsymbol{w} = \boldsymbol{w}_- = \boldsymbol{w}_+$ from bottom to top along the left or right edge; and $\boldsymbol{r}_0$ be the point at the center of the parallelogram as shown in the diagram.



a) Parametrize the surface of the parallelogram as $\boldsymbol{r}'(s,t)$, with the top and bottom edges are at $t = +\frac{1}{2}$ and $-\frac{1}{2}$, and the left and right edges are at $s = +\frac{1}{2}$ and $-\frac{1}{2}$ respectively.

b) Write down the Biot-Savart integral for the magnetic field in terms of integration parameters s, t and constants $\boldsymbol{\mathscr{r}}_0 \equiv \boldsymbol{r} - \boldsymbol{r}'_0$, $\boldsymbol{\ell}$, and $\boldsymbol{w}$.

c) Expand in powers of s and t, to calculate the integral up to second order.]

d) It is not possible to tile arbitrary surfaces with parallelograms—we need all four points on the quadrilateral to be arbitrary. To generalize this solution, let $\boldsymbol{\ell}_0$, $\boldsymbol{w}_0$ be the corresponding vectors through the center of the quadrilateral. We need one more vector $\boldsymbol{u}_0 = \boldsymbol{\ell}_+ - \boldsymbol{\ell}_- = \boldsymbol{w}_+ - \boldsymbol{w}_-$, where $\boldsymbol{\ell}_\pm$ run across the top and bottom, and $\boldsymbol{w}_\pm$ run along the right and left sides of the diagram. Generalize steps (a)-(c) to calculate $\boldsymbol{B}(\boldsymbol{r})$.]