University of Kentucky, Physics 306
Homework #11, Rev. A, due Wednesday, 2024-04-03

0. Potential theory—[*bonus*: the *Fundamental Theorem of Differentials (FTD)* $d\int_r \omega + \int_r d\omega = \omega$ implies that if $d\omega = 0$, then the *potential* $\alpha \equiv \int_r \omega \equiv \left[\int_{r'=0}^{r} \omega_{r\Omega}(r',\Omega)dr'\right] d\Omega$, integrated along radial coordinate lines, is its *antiderivative*: $d\alpha = \omega$. We use this formula to derive the antiderivative of the following special vector fields, generalizing the *Fundamental Theorem of Calculus (FTC i)*.

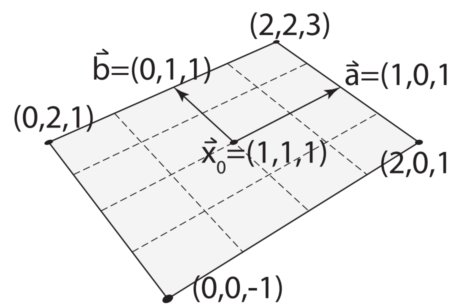
a) Show that if $\boldsymbol{\nabla} \times \boldsymbol{E} = \boldsymbol{0}$ (ie. $\boldsymbol{E}$ is *irrotational*), then $\boldsymbol{E} = -\boldsymbol{\nabla} V$ (ie. $\boldsymbol{E}$ is *conservative*), with the potential function $V(\boldsymbol{r}) = -\int_r \boldsymbol{E}\cdot \boldsymbol{dl} + C$, where C is any constant. Show that for a radial path in spherical coordinates, this reduces to $\boldsymbol{V} = -\boldsymbol{r} \cdot \int_0^1 \boldsymbol{E}(\lambda \boldsymbol{r}) d\lambda$. Confirm $\boldsymbol{E} = -\boldsymbol{\nabla} V$.

b) Show that if $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ (ie. $\boldsymbol{B}$ is *incompressible*), then $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ (ie. $\boldsymbol{B}$ is *solenoidal*), with the potential function $\boldsymbol{A} \cdot \boldsymbol{dl} = \int_r \boldsymbol{B} \cdot \boldsymbol{da} + d\chi$, where the 'constant of integration' $d\chi$ is the differential (gradient) of any scalar field. Show that for a radial path in spherical coordinates, this reduces to $\boldsymbol{A} = -\boldsymbol{r} \times \int_0^1 \boldsymbol{B}(\lambda \boldsymbol{r})\, \lambda\, d\lambda$. Take the curl to confirm $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$.

c) Show that since the differential of any density field $\rho(\boldsymbol{r})d\tau$ is zero, it can be written as the exact differential $\rho = \boldsymbol{\nabla} \cdot (\boldsymbol{D} + \nabla \times \boldsymbol{h})$ of some flux field $\boldsymbol{D}(\boldsymbol{r}) \cdot \boldsymbol{da}$, and a 'constant of integration' gauge $\boldsymbol{h}(\boldsymbol{r}) \cdot \boldsymbol{dl}$. Derive the formula for $\boldsymbol{D}$ for a radial path in spherical coordinates.]

1. Stokes' theorems—the FTD also generalizes the FTC (ii), integrating the derivative of a field over a region out to the boundary, resulting in an integral of the original field on the boundary.

a) Integrate ${}_{\mathcal{S}}\int \boldsymbol{\nabla} \times \boldsymbol{v} \cdot \boldsymbol{da}$ where $\boldsymbol{v} = \hat{\boldsymbol{x}}\, x^2 + \hat{\boldsymbol{y}}\, 2yz + \hat{\boldsymbol{z}}\, xy$, and $\mathcal{S}$ is the parallelogram in the figure to the right. Integrate ${}_{\partial S}\oint \boldsymbol{v} \cdot \boldsymbol{dl}$ along the boundary ∂S of $\mathcal{S}$ to verify Stokes' theorem.



b) Verify Stokes' theorem with the function in H10#1d) on the disk $\rho < R, z = 0$ in the xy-plane centered at the origin.

c) Verify Gauss' theorem with the function in H10#1e) inside the sphere $r < R$ of radius R centered at the origin.

2. Electrostatic and Magnetostatic integrals

a) Integrate i) $V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\oint \frac{\sigma da'}{\mathscr{r}}$, where $\hat{\boldsymbol{\mathscr{r}}}\mathscr{r} = \boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{r}'$, and ii) $\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\oint \frac{\sigma da' \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$ over $\boldsymbol{r}'$ on a sphere of radius R to calculate the electric potential V, field $\boldsymbol{E}$ of a spherical shell of uniform surface charge density σ. Verify that iii) $V = -\nabla \boldsymbol{E}$ and that you get the same iv) field from Gauss' law $\Phi_D = \oint_{\partial V} \epsilon_0 \boldsymbol{E} \cdot d\boldsymbol{a} = \int_V \rho d\tau = Q$ and v) potential $\boldsymbol{V}(\boldsymbol{r}) = -\int_{\boldsymbol{0}}^{\boldsymbol{r}} \boldsymbol{E} \cdot \boldsymbol{d\ell}$ from the FTVC.

b) Integrate i) the magnetic field $\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\oint \frac{I\boldsymbol{d\ell}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$ of a infinite wire on the z-axis and ii) show that you get the same field from Ampère's law/Stokes' theorem $\mathcal{E}_H = \oint_{\partial S} \boldsymbol{H} \cdot \boldsymbol{d\ell} = \int_S \boldsymbol{J} \cdot \boldsymbol{da} = I$, where $\boldsymbol{B} = \mu_0 \boldsymbol{H}$. Show that iii) $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, the curl of the vector potential $\boldsymbol{A} = -\frac{\mu_0 I}{2\pi}\hat{\boldsymbol{z}} \ln \rho$, and that iv) $\boldsymbol{H} = -\boldsymbol{\nabla} U$, the gradient of the scalar potential $U = -\frac{I}{2\pi}\phi$ with a discontinuity of $\Delta U = I$ at the *branch cut* $\phi = \pm\pi$, because $\boldsymbol{H}$ is not conservative—it has curl along the z-axis.

c) Expand the formula for the magnetic field $\boldsymbol{B}(\boldsymbol{r}) = \oint \frac{\mu_0}{4\pi}\frac{I\boldsymbol{d\ell}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$ of a ring of radius R centered in the xy-plane, and integrate $\boldsymbol{B}(z)$ on the z-axis. [*bonus*: integrate the field everywhere using elliptic functions, per www.grant-trebbin.com/2012/04/off-axis-magnetic-field-of-circular.html]