University of Kentucky, Physics 306
Homework #12, Rev. A, due Wednesday, 2023-04-10

0. Vector chain and product rules

a) [*bonus*: Prove i) $\nabla f \circ g = \nabla f(g(\boldsymbol{r})) = f'(g(\boldsymbol{r}))\nabla g(\boldsymbol{r})$; ii) $\nabla fg = g\nabla f + f\nabla g$; and iii) $\nabla \cdot f\boldsymbol{A} = \boldsymbol{A} \cdot \nabla f + f\nabla \cdot \boldsymbol{A}$. Classify and prove all product rules including $\nabla(\boldsymbol{A} \cdot \boldsymbol{B})$ and $\nabla \times (\boldsymbol{A} \times \boldsymbol{B})$.]

b) Calculate i) $\nabla(r^2 = \boldsymbol{r} \cdot \boldsymbol{r} = x^2 + y^2 + z^2)$; ii) $\nabla(r = \sqrt{r^2})$; iii) $\nabla \times \boldsymbol{r}$; and iv) $\nabla \cdot \boldsymbol{r}$.

c) Use the results of a) and b) to calculate i) ∇r^m; ii) $\nabla\, \boldsymbol{k} \cdot \boldsymbol{r}$; iii) $\nabla \times (\boldsymbol{k} \times \hat{\boldsymbol{r}} r^m)$ (compare with H10#1d); iv) $\nabla \cdot (k\hat{\boldsymbol{r}} r^m)$; v) $\nabla \cdot \hat{\boldsymbol{r}}/r^n$ (compare with H10#1e), where $\boldsymbol{k}$ is a constant vector.

1. Scalar Laplacian and inverse: Green's function

a) Combine the formulas for divergence and gradient to obtain the formula for $\nabla^2 f(\boldsymbol{r})$, called the *scalar Laplacian*, in orthogonal curvilinear coordinates (q^1, q^2, q^3) with scale factors h_1, h_2, h_3. Expand in Cartesian, cylindrical, and spherical coordinates.

b) Calculate $\nabla(1/r)$ and then $\nabla \cdot \nabla(1/r)$ using #0ci,ii), and compare with $\nabla^2(1/r)$ from #2a). At what point are the gradient, divergence, and Laplacian singular?

c) Use Gauss' theorem to show $\int_{r<a} d\tau \boldsymbol{\nabla} \cdot \hat{\boldsymbol{r}}/r^2 = 4\pi$ and thus $-\nabla^2(1/4\pi r) = \delta^3(\boldsymbol{r})$. We call $G(\boldsymbol{\mathscr{r}}) \equiv -\nabla^{-2}\delta^3(\boldsymbol{\mathscr{r}}) = 1/4\pi\mathscr{r}$ the *Green's function.* $\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{r}'$ shifts the *pole* from $\boldsymbol{r} = \boldsymbol{0}$ to $\boldsymbol{r}'$.

d) Solve for a *particular solution* of the second order partial differential equation $-\nabla^2 V(\boldsymbol{r}) = \rho(\boldsymbol{r})$ by expanding $\rho(\boldsymbol{r}) = \int d^3\boldsymbol{r}' \delta^3(\boldsymbol{\mathscr{r}})\rho(\boldsymbol{r}')$, and applying c) to each basis function $\delta^3(\boldsymbol{\mathscr{r}}) = \delta^3(\boldsymbol{r} - \boldsymbol{r}')$. Thus the Green's function can be used to invert the Laplacian operator acting on any function!

2. Vector Laplacian and decomposition: Helmholtz theorem

a) Write down all possible combinations of gradient, curl, and divergence to form second vector derivatives of both scalar and vector fields. Which 'natural' second derivatives are zero? Show that $\nabla^2 \boldsymbol{F} = \nabla\nabla \cdot \boldsymbol{F} - \nabla \times (\nabla \times \boldsymbol{F})$, where $\nabla^2 = \nabla \cdot \nabla = \partial_x^2 + \partial_y^2 + \partial_z^2$ (in Cartesian coordinates only) is the *vector Laplacian.* Thus the Laplacian, with *longitudinal* $\nabla\nabla \cdot \boldsymbol{F}$ and *transverse* $\nabla \times (\nabla \times \boldsymbol{F})$ components, is *the* unique second derivative of a vector field. In other coordinates, the Laplacian is defined in terms of curvilinear gradient, divergence and curls by the above equation. [*bonus*: Expand $\nabla^2 \boldsymbol{F}$ in terms of scale factors in an orthogonal curvilinear coordinate system.]

b) Use the technique of #1d) to solve the above equation for $\boldsymbol{F}$, and thus prove that any vector field $\boldsymbol{F}(\boldsymbol{r})$ can be decomposed into $\boldsymbol{F} = \boldsymbol{E} + \boldsymbol{B}$: an *irrotational* field $\nabla \times \boldsymbol{E} = \boldsymbol{0}$ (the *longitudinal* part), and an *incompressible* field $\nabla \cdot \boldsymbol{B} = 0$, (the *transverse* part) with *sources* $\rho = \nabla \cdot \boldsymbol{E}$ and $\boldsymbol{J} = \nabla \times \boldsymbol{B}$, respectively. Show that the vector source must satisfy $\nabla \cdot \boldsymbol{J} = \boldsymbol{0}$. Show that $\boldsymbol{E} = -\nabla V$ (it is *conservative*) and has potential $V = -\nabla^{-2}\rho$, and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ (it is *solenoidal*) and has potential $\boldsymbol{A} = -\nabla^{-2}\boldsymbol{J}$. Thus any vector field $\boldsymbol{F}$ with sources vanishing fast enough as $\boldsymbol{r} \to \infty$ is determined by its two sources $\rho = \nabla \cdot \boldsymbol{F}$ and $\boldsymbol{J} = \nabla \times \boldsymbol{F}$. The *Helmholtz theorem* also holds in finite regions after including boundary terms. Expand $-\nabla^{-2}$ as the integral of a Green's function.

c) [*bonus*: Given a field $\boldsymbol{F}$ with uniform sources ρ_0 and $\boldsymbol{J}_0$, calculate its potentials V and $\boldsymbol{A}$ and its longitudinal and transverse components $\boldsymbol{E}$ and $\boldsymbol{B}$. Note that this field diverges as $\boldsymbol{r} \to \infty$, but it is easier to calculate than other examples.]

3. [*bonus*: Five formulations of **Coulomb's** and **Ampere's laws**: the laws of *electrostatics* and *magnetostatics* have a twisted symmetry, in that they follow the same pattern, except the *electric field* $\boldsymbol{E}$ and *displacement* $\boldsymbol{D} = \epsilon\boldsymbol{E}$ are longitudinal, while the magnetic intensity $\boldsymbol{H}$ and flux $\boldsymbol{B} = \mu\boldsymbol{H}$ are transverse. This separation occurs because of the absense of *magnetic monopoles* (charge). The following *dervative chain* diagram shows the relation between electric/magnetic potentials, fields, and longitdinal/transverse sources of both fields:

$$\begin{tikzcd}
\chi \arrow[r, "d"] & (V,\vec{A}) \arrow[r, "d"] & (\vec{E},\vec{B}) \arrow[r, "d"] \arrow[d, shift right, "\varepsilon"'] \arrow[dr, "\sigma"] & 0 & \\
 & (\vec{C},I) \arrow[r, "d"] & (\vec{D},\vec{H}) \arrow[r, "d"] \arrow[u, shift right, "\mu"'] & (\rho,\vec{J}) \arrow[r, "d"] & 0
\end{tikzcd}$$

The integral and differential relations between each field, its sources, and potentials are classified into five formulations of electrostatics and magnetostatics, each formulation equivalent to Coulomb's or Ampere's law. Formulations II) and III) include two equations: one for the flux/divergence and one for the flow/curl. Likewise, formulations IV) and V) include two equations: one defining the potential and the other in terms of it.

Coulomb (I)

$$\vec{\boldsymbol{E}} = \int \frac{\text{đ}q'\,\hat{\mathscr{r}}}{4\pi\epsilon_0\,\mathscr{r}^2}$$

Helmholtz

$$\vec{\boldsymbol{F}} = q\vec{\boldsymbol{E}} \qquad W = qV$$

Integral field (II)

$$\Phi_E = Q/\epsilon_0 \qquad \mathcal{E}_E = 0 \ \text{(closed regions)}$$

Gauss / Stokes

Differential field (III)

$$\nabla\cdot\vec{\boldsymbol{E}} = \rho/\epsilon_0 \qquad \nabla\times\vec{\boldsymbol{E}} = \boldsymbol{0}$$

Potential (IV)

$$\mathcal{E}_E = -\Delta V \qquad V = \int \frac{\text{đ}q'}{4\pi\epsilon_0\,\mathscr{r}}$$

Poincaré / FTVC / Laplace Green

Poisson eq. (V)

$$\vec{\boldsymbol{E}} = -\nabla V \qquad \nabla^2 V = -\rho/\epsilon_0$$

Biot-Savart (I)

$$\vec{\boldsymbol{B}} = \int \frac{\mu 0}{4\pi}\,\frac{\vec{\boldsymbol{J}}\,d\tau\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$

Helmholtz

$$\vec{\boldsymbol{F}} = q\vec{\boldsymbol{v}}\times\vec{\boldsymbol{B}} \qquad \vec{\boldsymbol{p}} = q\vec{\boldsymbol{A}}$$

Integral field (II)

$$\mathcal{E}_B = \mu_0 I \qquad \Phi_B = 0 \ \text{(closed regions)}$$

Stokes / Gauss

Differential field (III)

$$\nabla\times\vec{\boldsymbol{B}} = \mu_0\vec{\boldsymbol{J}} \qquad \nabla\cdot\vec{\boldsymbol{B}} = 0$$

Potential (IV) ?

$$\vec{\boldsymbol{A}} = \int \frac{\mu_0}{4\pi}\,\frac{\vec{\boldsymbol{J}}\,d\tau}{\mathscr{r}}$$

Poincaré / ? / Laplace Green

Poisson (V)

$$\vec{\boldsymbol{B}} = \nabla\times\vec{\boldsymbol{A}} \qquad \nabla^2\vec{\boldsymbol{A}} = -\mu_0\vec{\boldsymbol{J}}$$

a) For each arrow in the above diagram, use the suggested theorem to show how the origin formulation leads to the destination one.

b) What is the integral formulation of $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$? Note the integral on both sides!

c) Find a sixth formulation VI) by applying the pattern IV) $\leftrightarrow$ V) to I) on the top row.]