University of Kentucky, Physics 306
Homework #14, Rev. A, Bonus

1. The Relaxation method, largely due to R. V. Southwell, is a straightforward method of solving PDEs based on the mean value property of the Laplacian. It is applicable to a wide variety of problems, not only Laplace's equation. We will use it to calculate the potential distribution within a square with two adjacent edges at 100 V, one at 0 V and one at 50 V.

a) Draw a square grid of 36 points: 20 on the edges of the square, and 16 in the interior.

b) Fill in the boundary potentials on all edges, and initial estimates of the potentials at the interior points on your grid. This is your first approximation.

c) Correct your first approximation as follows: replace the value of each interior point with the average of its four adjacent squares. Explain why this procedure 'relaxes' the potential closer to $\nabla^2 V = 0$ (see #2b).

d) Repeat part c) until none of the changes are greater than 0.5 V, and sketch the equipotentials for 20 V, 40 V, 60 V, and 80 V. You may want to solve this problem on a computer, using Excel, Matlab, or Octave (open source program similar to Matlab).

How would one extend this method to solve Neumann (flux) boundary conditions?

2. Finite Differences Method. Re-solve problem #1 noniteratively as outlined below:

a) Given the discrete approximation of a function $f_i = f(x_i)$ evaluated at regular intervals $x_i = x_0 + i\,\Delta x$, show that $f'(x_i) \approx (f_{i+1} - f_i)/\Delta x \approx (f_i - f_{i-1})/\Delta x$ are two approximations of the first derivative. What is their average?

b) Derive a formula for the discrete second derivative using the two first derivatives of part a), and extend it to a discrete 2-dimensional Laplacian.

c) Turn Laplace's equation into a matrix equation by treating the function $V(x)$ on interior points of the grid in problem #1 as a 16-component vector $\boldsymbol{V} = [V_{11}, V_{12}, V_{13}, V_{14}, V_{21}, \ldots, V_{44}]$. Use part b) to represent $\nabla^2 V$ as a 16×16 matrix $\boldsymbol{K}$ multiplied by $\boldsymbol{V}$. Note that the Laplacian can be approximated by a matrix because it is a linear operator. Each row should represent Laplace's equation evaluated at one interior point. There is no place in $\boldsymbol{KV}$ for the fixed boundary potentials V_{0n}, V_{5n}, V_{m0}, and V_{m5} for $m, n = 1, 2, 3, 4$, so put these values in a fixed vector $\boldsymbol{B}$, such that the matrix equation $\boldsymbol{KV} + \boldsymbol{B} = \boldsymbol{0}$ represents the full boundary value problem.

d) Solve $\boldsymbol{KV} = -\boldsymbol{B}$ for the potential V at each point and compare your result with #1. DO NOT do this by hand! Try using the online applet http://www.bluebit.gr/matrix-calculator or one of the software packages Octave, Matlab, Maple, or Mathematica.

3. Finite Element Method. In this problem we will investigate one of the most common methods of solving partial differential equations numerically (especially involving the Laplacian). This method is very flexible and can be used to solve PDEs on irregularly shaped domains (like cars, or electrodes for your awesome new experiment). This problem is self-contained, but for extra details or hints, refer to the article http://wikipedia.org/wiki/Finite_element.

a) Show that

$$\int_R (\nabla^2 u) v d\tau = \oint_{\partial R} (\nabla u) v \cdot \boldsymbol{da} - \int_R \nabla u \cdot \nabla v d\tau. \tag{1}$$

Assuming that $v = 0$ on the boundary, this means that the equation $\nabla^2 u = f$ can be written

$$-\int_R \nabla u \cdot \nabla v d\tau = \int_R f v d\tau, \tag{2}$$

which is now a first order integral equation which must hold true for any *test function* $v(\boldsymbol{r})$. This is called the *weak form* of Laplace's equation.

b) In a one-dimensional space, Eq. 2 can be discretized by choosing appropriate "basis functions" for $v(\boldsymbol{r})$. The i^{th} *tent function* is defined as $v_i(x) = (1 - |x - i|)\, \theta(1 - |x - i|)$, where $\theta(x) = \{1$ if $x > 0$, and 0 if $x < 0\}$ is the Heaviside step function and $i = 1, 2, 3, 4$. Plot each of these functions on the same graph.

c) Sketch the function $f(x) = 2v_1(x) + 4v_2(x) + 3v_3(x) + 1v_4(x)$. In the same way, any function defined on $0 < x < 5$ can be approximated by a linear combination of these basis functions $f(x) \approx \sum_{i=1}^{4} f_i v_i(x)$, where $f_i = f(i)$.

d) Convert Eq. 2 into four linear equations ($i = 1, 2, 3, 4$) by substituting $v(x) \to v_i(x)$ and approximating $u(x) = \sum_j u_j v_j(x)$ and $f(x) = \sum_j f_j v_j(x)$. Combine these equations into a single matrix equation $-Lu = Mf$, where $u = [u_1, u_2, u_3, u_4]^T$, $v = [v_1, v_2, v_3, v_4]^T$, $M_{ij} = \int_{-\infty}^{\infty} v_i(x) v_j(x) dx$ and $L_{ij} = \int_{-\infty}^{\infty} \nabla v_i(x) \cdot \nabla v_j(x) dx$. Note that $\nabla = d/dx$ in one dimension. Perform the integrals of each component of L and M to obtain numerical matrices.

e) Use part d) to solve the boundary value problem $\nabla^2 u = 3$ on the region $0 < x < 5$, with boundary conditions $u(0) = 0$ and $u(5) = 0$, by solving the above matrix equation for u_i. Solve the boundary value problem analytically and compare with the finite element result.

Describe how this method could apply to higher dimensions. Hint: Define the 2-dimensional tent function $v_{ij}(x, y) = v_i(x)\, v_j(y)$, and show how it can be used as above. The matrices get very large for 2- or 3-dimensional problems, but there are packages like FlexPDE (http://www.pdesolutions.com, free student version) and COMSOL (http://comsol.com) which do all the bookkeeping, and implement this method for arbitrary geometries and complex partial differential equations from a GUI interface.