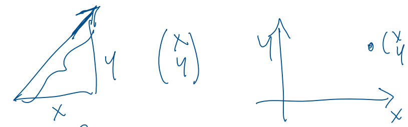
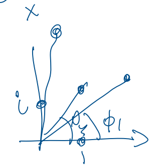
L04 Vectors

Sunday, January 21, 2024 9:19 PM

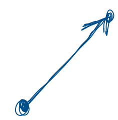
- The ***utility of vectors*** comes from their ***geometric interpretation***, which describes many concepts in physics; however the ***power of vectors*** comes from their ***algebraic structure***, which allows for analytic calculations and applications beyond spatial directions.



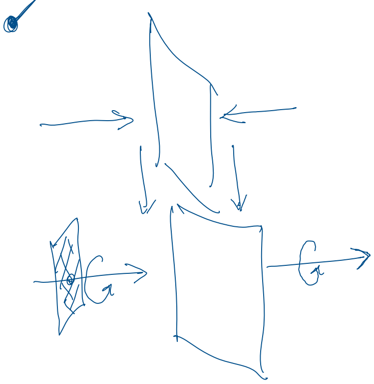
  - Rene Descartes pioneered this dichotomy with the introduction of ***analytic geometry*** over 400 years ago in 1619, by describing points in Cartesian coordinates.
  - We have already seen the power in the geometric/algebraic connections of addition/multiplication/conjugation of points in the complex plane to describe reflection/stretches/rotation. This is a good example of an algebraic vector space.



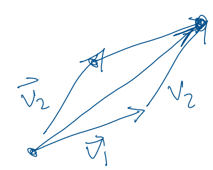
  - This lecture will treat the specific structure associated with plain vector spaces, which we later augment with other multilinear structure:
    1) dot/cross/triple products, 2) linear operators, 3) dual vectors, etc.



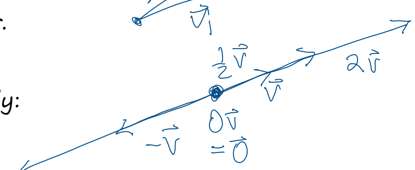
- In Physics I/II, vectors introduced geometrically as ***arrows*** with lengths, directed in space.
  - The vector arrow has a ***tail*** and a ***head*** a certain ***displacement*** from the tail, which is what it represents--it is the same vector no matter where it is positioned in space.
  - It has a magnitude in some units (often but not always involving length) and a physical direction representing the displacement in position $d\vec{r}$, velocity (or momentum $\vec{p} = m\vec{v}$), acceleration (or force $\vec{F} = m\vec{a}$, an example involving direction without displacement).
  - All of these are called ***polar vectors*** because they imply force or movement in direction of the arrow, which is the opposite direction reflected out from (but not along) a mirror.
  - When we talk about the cross product, we will discover ***axial vectors***, which represent rotation about an axis, or the area perpendicular to a normal, which has the opposite properties under reflection, and should not be added with polar vectors (***parity violation***)



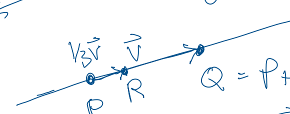
- There are only two operations we can do with pure vectors:
  1) Add two vectors by combining the two head-to-tail to get the total displacement
     Combining them in either way yields the same result: the far corner of a parallelogram.
  2) Scaling a vector by 1 (unchanged), -1 (reflected), 2 (doubled) 1/2 (halved),
     0: the unique vector with no length and thus arbitrary direction, or any real number.



- Moving a vector around is not an operation because it does not change it
  - This distinguishes ***vectors*** from ***points***, which Descartes originally described analytically: Points have a fixed location, while vectors can be moved anywhere without changing.



  - Some operations between points and vectors make sense:
    - The sum of a point and vector displaces the point from P to $Q = P + \vec{v}$
    - The difference between two points $\vec{v} = Q - P$ is the displacement between them
    - The point $R = P + \alpha\vec{v} = (1 - \alpha)P + \alpha Q = \alpha Q + \beta P$, $\alpha + \beta = 1$, a fraction α of the distance from P to Q. This universal ***affine combination*** defines the structure of points in an ***affine space***, which we will explore in optional H02#3.



$Q = P + \vec{v}$

$Q - P = \vec{v}$

head − tail

$R = P + \frac{1}{3}\underbrace{\vec{v}}_{Q-P}$

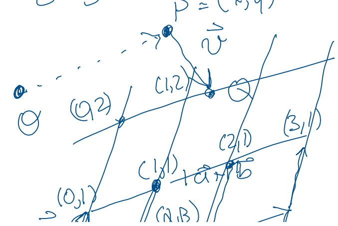
$= P + \frac{1}{3}Q - \frac{1}{3}P$

$= \frac{2}{3}P + \frac{1}{3}Q$

$\frac{2}{3} + \frac{1}{3} = 1$

  - If we represent points by the vectors from a common ***origin*** (tail) to the points (head), then all operations on points/vectors are reduced to the corresponding vector operations. These are called ***position vectors***, and are drawn simply as points to reduce clutter.
  - There are key differences, though, especially in curvilinear coordinate systems.
- The analog of affine combinations of points is the universal ***linear combination*** of vectors $\vec{v} = \vec{a}\alpha + \vec{b}\beta + \vec{c}\gamma + \cdots$, which encompases both addition and scalar multiplication.
  - Vectors obey all algebraic properties of addition and multiplication like ***distribution*** except the product $\vec{u}\vec{v}$ is not defined without extra structure
  - These rules guarantee that any algebraic formula involving vectors can be reduced to an affine combination of distinct vectors $\vec{a}, \vec{b}, \vec{c}, \ldots$
  - The set of all linear combinations for coefficients $\alpha, \beta, \gamma \in \mathbb{R}$ is a ***linear [vector] space***.
    It is ***closed***—it includes all possible linear combinations of vectors in the space.

$P = (x, y)$



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- The set of all linear combinations for coefficients $\alpha, \beta, \gamma \in \mathbb{R}$ is a ***linear [vector] space***. It is ***closed***--it includes all possible linear combinations of vectors in the space.

- A vector space can be extended or restricted, while still being closed
  - Any set of linear combinations of 1,2,3 or more vectors for all coefficients is closed.
  - For example, the line of points (position vectors) through the origin $V_1 = \{\vec{v}_1 = \vec{a}\alpha | \alpha \in \mathbb{R}\}$ and the plane $V_2 = \left\{\vec{v}_2 = \vec{b}\beta + \vec{c}\gamma | \beta, \gamma \in \mathbb{R}\right\}$ ***spanned*** by vectors $\vec{b}$ and $\vec{c}$ are ***subspaces*** of all 3d vectors (subsets which are vector spaces in their own right).
  - All subspaces have the $\vec{0}$ vector in common. If that is the only one, then one can form a ***direct sum*** $V = V_1 \oplus V_2$ of the two spaces, from the unique sum of one vector in each subspace, for example $\vec{v} = \vec{v}_1 + \vec{v}_2$ in $V_1 = \{\vec{a}\alpha | \alpha \in \mathbb{R}\}$ and $V_2 = \left\{\vec{b}\beta + \vec{c}\gamma | \beta, \gamma \in \mathbb{R}\right\}$ respectively.
  - Conversely, $\vec{v}_1 = \vec{a}\alpha$ and $\vec{v}_2 = \vec{b}\beta + \vec{c}\gamma$ are ***projections*** of $\vec{v}$ into the ***complementary*** spaces V_1 and V_2. You need both spaces to determine the projection: from the head of $\vec{v}$, you slide in the direction of $\vec{a}$ till you land on the unique vector $\vec{v}_2 = \vec{b}\beta + \vec{c}\gamma \in V_2$, or in the directions of $\vec{b}$ and $\vec{c}$ till you get to $\vec{v}_1 = \vec{a}\alpha \in V_1$. The sum of the projections is $\vec{v} = \vec{v}_1 + \vec{v}_2$.
  - Projections are the shadow of a vector, with light shining in one direction onto another
  - Projections are in inverse of Linear Combinations: LC builds up, while P decomposes
  - We need the notion of a dot product to define ***orthogonality*** (perpendicular) before we can talk about the parallel/perpendicular projection along a single vector $\vec{a}$.

- Linear combinations projecting $\vec{v}$ to the unique sum $\vec{v} = \vec{a}\alpha + \vec{b}\beta + \vec{c}\gamma + \cdots$ with coefficients $\mathbb{v} = \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$ and fixed vectors $\vec{\mathbb{b}} = (\vec{a} \quad \vec{b} \quad \vec{c})$, treat geometric vectors algebraically à la Descartes.
  - The vectors $\vec{\mathbb{b}}$ are called called a ***basis*** of the vector space and the coefficients $\mathbb{v}$ are called the ***components*** of the vector $\vec{v}$ with respect to the basis $\vec{\mathbb{b}}$.
  - Components of position vectors are equal to the Cartesian coordinates of points, but coordinates (ie. cylindrical or spherical) do not have to be linear like components do.
  - Each vector is identified and manipulated algebraically as a tuple of real numbers $\mathbb{v} \in \mathbb{R}^n$.
  - For unique existence of this correspondence, a basis has the two properties of projections:
  1) It is ***independent***: if $\vec{v} = \vec{a}\alpha + \vec{b}\beta + \vec{c}\gamma = \vec{0}$, then $\alpha = \beta = \gamma = 0$.
  2) It is ***complete*** or ***spans*** the space: any vector can be written $\vec{v} = \vec{a}\alpha + \vec{b}\beta + \vec{c}\gamma$.

EXAMPLES:

1) An example of a vector space, subspaces, basis, and components is Complex numbers $V = \{z = x + iy | x, y \in \mathbb{R}\}$, with explicit linear combinations $z = 1x + iy$.
   - The basis is $\vec{\mathbb{b}} = (1 \quad i)$, and components of z are $\mathbb{v} = \binom{x}{y}$. Note that both are numbers!
   - The two subspaces are the real $V_1 = \{1x | x \in \mathbb{R}\}$ and imaginary $V_1 = \{iy | y \in \mathbb{R}\}$ number lines.
   - The space of complex numbers is the direct sum (grid) of these two lines.
   - A vector $z = 1x + iy$ projects onto the vectors $v_1 = 1x$ and $v_2 = iy$.
   - The basis vectors have canonical components $1 = \binom{1}{0}$ and $i = \binom{0}{1}$.
   - The only common vector is $\vec{0} = 0 = 0 + i0$.
2) A universal example is the space $\mathbb{R}^2$ with the canonical basis $\left\{\vec{e}_1 = \binom{1}{0}, \quad \vec{e}_2 = \binom{0}{1}\right\}$.

   In case, the vectors are indistinguishable their components.
   - This vector space is used for algebraic manipulation of any other 2-dimensional space.
3) The final ***isomorphic*** example is the set of arrows in the plane with basis arrows $\{\hat{x}, \hat{y}\}$.
   - In all three cases, the components are: $\mathbb{v} = \binom{x}{y}$

- The use of ***matrices*** collect the basis vector (column $\vec{\mathbb{b}}$) and compoents (row $\mathbb{v}$) to one object
  - I use the notation $\mathbb{a}$ to represent matrices, and $\vec{v}$ for vectors; $\vec{\mathbb{b}}$ is a matrix of vectors

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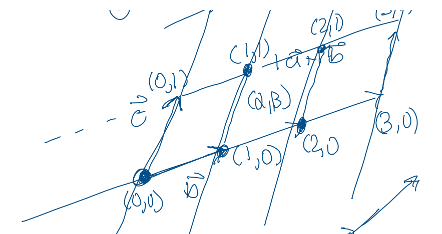
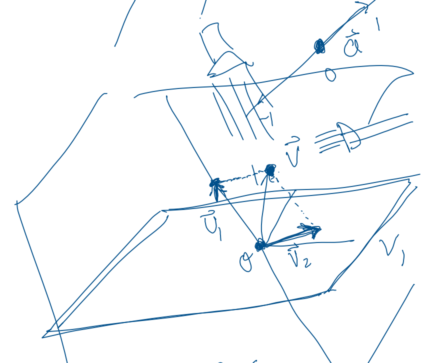
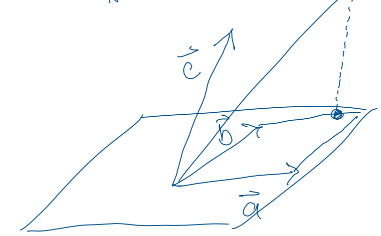




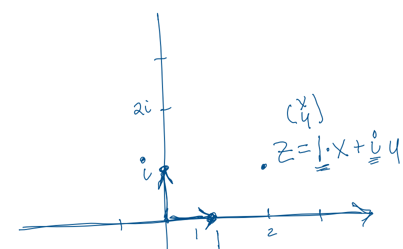
$V = V_1 \oplus V_2$

$PP\vec{v} = P\vec{v}$

$P^2 = P$



$\vec{c} - \alpha\vec{a} - \beta\vec{b} = \vec{0}$



$\mathbb{R} = \{1x \mid x \in \mathbb{R}\}$

$\mathbb{I} = \{iy \mid y \in \mathbb{R}\}$

$z = x + iy$

$z^* = x - iy$

$z + z^* = 2x \quad /2 \quad x = \frac{1}{2}(z + z^*)$

$z - z^* = 2iy \quad y = \frac{1}{2i}(z - z^*)$

$\binom{x}{y} = \binom{1}{0}x + \binom{0}{1}y$

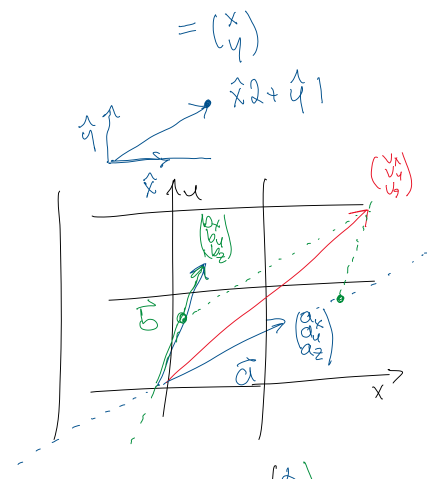
$= \binom{x}{0} + \binom{0}{y}$

$= \binom{x}{y}$

$\hat{x}2 + \hat{y}1$

- The use of ***matrices*** collect the basis vector (column $\vec{\mathbb{b}}$) and compoents (row $\mathbb{v}$) to one object
  - I use the notation $\mathbb{a}$ to represent matrices, and $\vec{v}$ for vectors; $\vec{\mathbb{b}}$ is a matrix of vectors
  - The ***contraction*** $\vec{v} = \vec{\mathbb{b}}\mathbb{v} = (\vec{a}\quad \vec{b}\quad \vec{c})\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \vec{a}\alpha + \vec{b}\beta + \vec{c}\gamma$ captures a ubiquitous patern of operations in linear algebra: 1) linear combinations, 2) dot products, 3) linear operators
    - Always between a left row and a right column of the same length.
  - Matrix equations can be ***augmented*** with extra rows on the left or columns on the right
    - Represents multiple equations with repetition: $(\vec{v}\quad \vec{w}) = \vec{\mathbb{b}}(\mathbb{v}\quad \mathbb{w}) = (\vec{a}\quad \vec{b}\quad \vec{c})\begin{pmatrix}\alpha & \delta\\ \beta & \epsilon\\ \gamma & \zeta\end{pmatrix}$ for both $\vec{v} = \vec{a}\alpha + \vec{b}\beta + \vec{c}\gamma$ and $\vec{w} = \vec{a}\delta + \vec{b}\epsilon + \vec{c}\zeta$, with the basis repeated
  - Augment vectors with components $\begin{pmatrix}v_x\\ v_y\\ v_z\end{pmatrix} = \begin{pmatrix}a_x & b_x & c_x\\ a_y & b_y & c_y\\ a_z & b_z & c_z\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$ to solve for α, β, γ
    - Or even solve for $\mathbb{v}$ and $\mathbb{w}$ at the same time!
  - Matrix multiplication is the matrix of all possible row-column contractions
  - We will learn more about matrices in each of the next few lectures

- ***Index notation***: complementary for same pattern $\vec{v} = \vec{b}_i\, v_i \equiv \sum_{i=1}^{3} \vec{b}_i v_i = \vec{b}_1 v_1 + \vec{b}_2 v_2 + \vec{b}_3 v_3$
  - Summation is implied for repeated index i (sometimes called Einstein notation).
  - It still works in complicated situations like 3d (cube) matrices with many contractions
  - We will also learn more about index notation in the next few lectures



$\vec{v} = (\vec{a}\ \vec{b})\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$

$\begin{pmatrix}v_x\\ v_y\end{pmatrix} = \underbrace{\begin{pmatrix}a_x & b_x\\ a_y & b_y\end{pmatrix}}_{\text{known}}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$

$\mathbb{V}' = \mathbb{B}\,\underline{\underline{\mathbb{V}}}$

$\mathbb{B}^{-1}\mathbb{V} = \mathbb{V}$