L05 Inner/Outer Products

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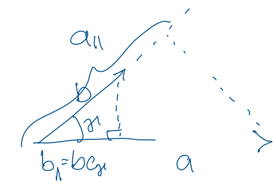
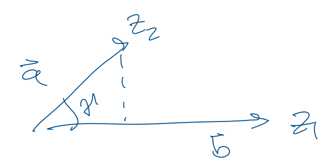


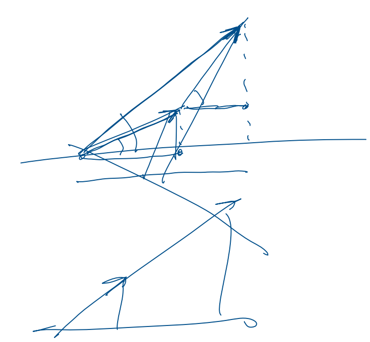
- The linear structure of L04 entailed parallel lines, but NOT explicit angle/length
  - Projections needed two complementary spaces: along and into
  - Even $\hat{x}$, $\hat{y}$, $\hat{z}$ had no dependence on unit length or perpendicularity (just implied).
  - The dot product provides notions for all of these (unit vectors, orthogonal projections)
- Like vectors themselves, the ***dot product*** has geometric significance and analytic power
  - Connection between these two aspects is again the linear structure of the dot product
  - Definition: $\vec{a}\cdot\vec{b} = ab\cos\gamma = a_\parallel b = ab_\parallel = a_x b_x + a_y b_y + a_z b_z = \mathbb{a}^\mathrm{T}\mathbb{b} = \delta_{ij} a_i b_j$ $\quad c_\gamma \equiv \cos\gamma$

    and $\vec{a} = \hat{a}a = \hat{x}a_x + \hat{y}a_y + \hat{z}a_z = (\hat{x}\ \hat{y}\ \hat{z})\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \hat{\mathbb{e}}\mathbb{a} = \hat{e}_i a_i$; similarly for $\vec{b}$; and $\gamma = \angle(ab)$.
  - Length of $\vec{a}$ is $a = \sqrt{\vec{a}\cdot\vec{a}}$ and direction is the unit vector $\hat{a} = \vec{a}/a$, since $\cos 0° = 1 \to \vec{a}\cdot\vec{a} = a^2$.
- Geometry: dot product is ***product of parallel projected lengths*** $\vec{a}\cdot\vec{b} = a_\parallel b = ab_\parallel$.
  - Either $a_\parallel = a\cos\gamma$ ($\vec{a}$ onto $\vec{b}$) or $b_\parallel = b\cos\gamma$ ($\vec{b}$ onto $\vec{a}$), but not both onto some $\vec{n}$ (see below).
  - The projected product $a_\parallel b$ is not the projected length $a_\parallel$ unless $\vec{b} = \hat{b}$ is a unit vector.
  - ***projection***: $P_\parallel \vec{v} \equiv \vec{v}_\parallel = \hat{n}v_\parallel = \hat{n}\,\hat{n}\cdot\vec{v}$ thus $P_\parallel = \hat{n}\,\hat{n}\,\cdot$ is an ***operator*** on vectors $P_\parallel : \vec{v} \mapsto \vec{v}_\parallel$.
    1. Definition of projection: $P^2 = P$ because $P_\parallel^2 = \hat{n}\,\hat{n}\cdot(\hat{n}\,\hat{n}\,\cdot = \hat{n}\,(\hat{n}\cdot\hat{n})\,\hat{n}\,\cdot = \hat{n}\,\hat{n}\,\cdot = P_\parallel$
    2. Orthogonal projection: $P^T = P$ because $P_\parallel^T = \left(\hat{n}\,\hat{n}^T\right)^T = \hat{n}\,\hat{n}\,\cdot = P_\parallel$ (see below)



- The defining properties of the dot product relate the geometric → algebraic calculation:
  1) ***scalar*** valued: (product of lengths), polar · polar, axial · axial mirror-invariant
  2) ***symmetric*** (commutative): $\vec{a}\cdot\vec{b} = ab\cos\gamma = ba\cos(-\gamma) = \vec{b}\cdot\vec{a}$
  3) ***bilinear*** (distributive): $(\vec{a}_i\alpha_i)\cdot\left(\vec{b}_j\beta_j\right) = \alpha_i\left(\vec{a}_i\cdot\vec{b}_j\right)\beta_j = \alpha_i\, g_{ij}\, \beta_j$, where g_{ij} is the ***metric tensor***.
     This is a direct property of projections, head-to-tail addition, and similar triangles
  - Example - law of cosines: the length^2 of $\vec{c} = \vec{a} - \vec{b}$ is $c^2 = a^2 - 2\vec{a}\cdot\vec{b} + b^2 = a^2 + b^2 - 2ab\cos\gamma$

    In the complex plane, the vector is $\quad |c|^2 = c^*c = (a-b)^*(a-b) = |a|^2 + |b|^2 - 2Re(a^*b)$



- An ***orthonormal*** basis $\hat{\mathbb{e}} = (\hat{x}, \hat{y}, \hat{z})$, where $\hat{x}\cdot\hat{x} = 1$, $\hat{x}\cdot\hat{y} = 0$, etc., forms a unit cube.
  - Components are easy to calculate in this basis: $v_x = \hat{x}\cdot\vec{v}$, $v_y = \hat{y}\cdot\vec{v}$, $v_z = \hat{z}\cdot\vec{v}$.

    This is trivial in $\hat{\mathbb{e}} = (\hat{x}, \hat{y}, \hat{z})$, but powerful for a rotated orthonormal basis.
  - Dot product also simplifies: $\vec{v}\cdot\vec{w} = \left(\hat{x}v_x + \hat{y}v_y\right)\cdot\left(\hat{x}w_x + \hat{y}w_y\right) = v_x w_x + v_y w_y = \mathbb{v}^\mathrm{T}\mathbb{w} = v_i w_i$

    Thus in an orthonormal basis, $\vec{v}\cdot\vec{w}$ is a sum of projected products. If $\vec{v}$ has only one component, ie. $\vec{v} = \hat{x}v$, this reduces to the geometric interpretation $\vec{v}\cdot\vec{w} = vw_x$.
- Matrix notation - new concepts we need to treat the dot product.
  - ***Transpose*** M^T flips a matrix M along its main diagonal: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.
    - It is almost invariably connected with the dot product as in the contraction $\mathbb{v}^\mathrm{T}\mathbb{w}$.
    - Symmetry: M is ***symmetric*** if $M^T = M$ and ***antisymmetric*** if $M^T = -M$.
  - ***Hermitian conjugate*** $M^\dagger = M^{*T}$ also conjugates complex vectors: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$.
    - The transpose ignores scalars $(a)^T = (a)$ and complex conjugate applies to each element.
    - The inverse M^{-1} and transposes $M^T, M^\dagger$ all share a key property: $(AB)^T = B^T A^T$.
    - They are also all ***involutions*** (self-inverse): $\left(A^T\right)^T = A$.
  - ***Inner product***: $\mathbb{v}^\mathrm{T}\mathbb{w} = (v_x \quad v_y)\begin{pmatrix} w_x \\ w_y \end{pmatrix}$ is a contraction. Thus $\left(\mathbb{v}^\mathrm{T}\mathbb{w}\right)^\mathrm{T} = (\mathbb{w})^T\left(\mathbb{v}^\mathrm{T}\right)^\mathrm{T} = \mathbb{w}^\mathrm{T}\mathbb{v} = \mathbb{v}^\mathrm{T}\mathbb{w}$.
  - ***Outer product***: $\mathbb{v}\mathbb{w}^\mathrm{T} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}(w_x \quad w_y) = \begin{pmatrix} v_x w_x & v_x w_y \\ v_y w_x & v_y w_y \end{pmatrix}$, components of projection (above).
  - ***Identity matrix***: $\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has three(!) interpretations related to the dot product:
    - Multiplicative identity operator: $\mathbb{I}\mathbb{v} = \mathbb{v}$, $\mathbb{v}^\mathrm{T}\mathbb{I} = \mathbb{v}^\mathrm{T}$, $\mathbb{v}^\mathrm{T}\mathbb{I}\mathbb{w} = \mathbb{v}^\mathrm{T}\mathbb{w}$.
    - Canonical basis transformation: $\hat{\mathbb{e}} = (\hat{\mathbf{e}}_1 \quad \hat{\mathbf{e}}_2) = \mathbb{I}$ where $\hat{\mathbf{e}}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\hat{\mathbf{e}}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
    - Orthonormal metric: $\hat{\mathbb{e}}^\mathrm{T}\cdot\hat{\mathbb{e}} = (\hat{\mathbf{e}}_1 \quad \hat{\mathbf{e}}_2)^T\cdot(\hat{\mathbf{e}}_1 \quad \hat{\mathbf{e}}_2) = \begin{pmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \end{pmatrix}\cdot(\hat{\mathbf{e}}_1 \quad \hat{\mathbf{e}}_2) = \begin{pmatrix} \hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_2 \end{pmatrix} = \mathbb{I}$.

  We can use these properties to derive the inner product form of the dot product:

$\vec{v}\cdot\vec{w} = (\widehat{\mathbb{e}}\mathbb{v})\cdot(\widehat{\mathbb{e}}\mathbb{w}) = (\widehat{\mathbb{e}}\mathbb{v})^T\cdot(\widehat{\mathbb{e}}\mathbb{w}) = \mathbb{v}^{\mathrm{T}}\left(\widehat{\mathbb{e}}^T\cdot\widehat{\mathbb{e}}\right)\mathbb{w} = \mathbb{v}^{\mathrm{T}}\mathbb{I}\mathbb{w} = \mathbb{v}^{\mathrm{T}}\mathbb{w} = v_x w_x + v_y w_y$ *or* $\cdot \to {}^T$

The same formalism applies to the metric of a non-orthonormal basis:

$\vec{v}\cdot\vec{w} = (\vec{\mathbb{e}}\mathbb{v})\cdot(\vec{\mathbb{e}}\mathbb{w}) = (\vec{\mathbb{e}}\mathbb{v})^T\cdot(\vec{\mathbb{e}}\mathbb{w}) = \mathbb{v}^{\mathrm{T}}\left(\vec{\mathbb{e}}^T\cdot\vec{\mathbb{e}}\right)\mathbb{w} = \mathbb{v}^{\mathrm{T}}\mathbb{g}\mathbb{w} = \begin{pmatrix} v_x & v_y\end{pmatrix}\begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy}\end{pmatrix}\begin{pmatrix} w_x \\ w_y\end{pmatrix}$, $\cdot \to {}^T\,\mathbb{g}$

where the metric $\mathbb{g} = \vec{\mathbb{e}}^T\cdot\vec{\mathbb{e}} = \begin{pmatrix}\vec{\boldsymbol{e}}_1\cdot\vec{\boldsymbol{e}}_1 & \vec{\boldsymbol{e}}_1\cdot\vec{\boldsymbol{e}}_2 \\ \vec{\boldsymbol{e}}_2\cdot\vec{\boldsymbol{e}}_1 & \vec{\boldsymbol{e}}_2\cdot\vec{\boldsymbol{e}}_2\end{pmatrix} = \begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy}\end{pmatrix}$ *characterizes the dot product.*

- *Example: in statistics the multivariable extension of* $z = \frac{x-\mu}{\sigma}$ *is the residual* $\vec{\chi} = \vec{x} - \vec{\mu}$.
  To generalize σ, *we square* σ *to obtain the symmetric covariance matrix* $\Sigma = \begin{pmatrix}\sigma_{11}^2 & \sigma_{12}^2 \\ \sigma_{21}^2 & \sigma_{22}^2\end{pmatrix}$,
  which appears in $\chi^2 = \vec{\chi}\cdot\vec{\chi} = \chi^T W \chi$, *as the weight* $W = \Sigma^{-1}$ *called the* [*Mahalanobis metric*].

- *Index (implicit summation or Einstein) notation*
  - *Indexed elements of a matrix:* $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{pmatrix} \sim m_{ij}$ *always row,col same as contractions*
  - *Contraction:* $\vec{v}\cdot\vec{w} = \sum_i v_i w_i = v_i w_i$ *where the summation is implied (since it's always there!)*
  - *Augmentation:* $C = AB$ *or* $\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}$ *or* $c_{ik} = \sum_j a_{ij} b_{jk} = a_{ij} b_{jk}$,
    which represents 4 equations, one for each i *and* j, *for example* $c_{12} = a_{1j} b_{j2} = a_{11} b_{12} + a_{12} b_{22}$,
    representing a contraction for each row of A *and column of* B, *all augmented together.*
    The j *is called a 'dummy index'. It can be replaced by any symbol and expands to a sum.*
    Be careful to use a different dummy index for each new summation (only 2 i*'s or* j*'s).*
  - *The* ***Kronecker delta*** *symbol* $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j\end{cases}$ *represents the components of the identity matrix*
    $\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} = \begin{pmatrix}\delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22}\end{pmatrix}$, *and has the same property:* $\mathbb{I}\mathbb{v} = \mathbb{v}$ *or* $\delta_{ij} v_j = v_i$ *. Thus the rule to simplify equations is to remove* δ_{ij} *and replace all of one dummy index with the other. If both* i *and* j *are repeated, like* $\alpha = v_i \delta_{ij} w_j$, *you can replace either to get* $\alpha = v_i w_i = v_j w_j$.
  - *Conservation law of symbols: everything must match on the LHS and RHS of an equation:*
    1) units, 2) tensor rank (scalar, vector, operator, …), 3) matrix dimension, 4) indices

With these rules, we can repeat the calculation of dot product in terms of components:

$\vec{v}\cdot\vec{w} = (\hat{\boldsymbol{e}}_i v_i)\cdot\left(\hat{\boldsymbol{e}}_j w_j\right) = v_i\left(\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_j\right) w_j = v_i \delta_{ij} w_j = v_i w_i = v_x w_x + v_y w_y$.

The same formalism applies to the metric of a non-orthonormal basis:

$\vec{v}\cdot\vec{w} = (\vec{\boldsymbol{e}}_i v_i)\cdot\left(\vec{\boldsymbol{e}}_j w_j\right) = v_i\left(\vec{\boldsymbol{e}}_i\cdot\vec{\boldsymbol{e}}_j\right) w_j = v_i g_{ij} w_j = v_x g_{xx} w_x + v_x g_{xy} w_y + v_y g_{yx} w_x + v_y g_{yy} w_y$.

The indices 'run over' either 1,2,3 *or* x, y, z *for the labels of the different components.*