L06 Exterior Product

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- The ***cross product*** complements the linear structure of vectors and length of the dot product with area and rotation. It is also used for a projections complementary of the dot product. Thus this lecture is complementary with L05 - it is useful to compare and contrast them.
  - Definition: $\vec{a}\cdot\vec{b} = \hat{n}ab\sin\gamma = \hat{n}a_\perp b = \hat{n}ab_\perp = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_y \\ b_x & b_y & b_z \end{vmatrix} = \begin{vmatrix} \hat{\mathbb{e}} \\ \mathbb{a}^T \\ \mathbb{b}^T \end{vmatrix} = \hat{\varepsilon}_{ij}a_ib_j$ $\quad s_\gamma \equiv \sin\gamma$

    and $\vec{a} = \hat{a}a = \hat{x}a_x + \hat{y}a_y + \hat{z}a_y = (\hat{x}\ \hat{y}\ \hat{z})\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \hat{\mathbb{e}}\mathbb{a} = \hat{e}_ia_i$, similarly for $\vec{b}$, and $\hat{n}\gamma = \vec{Z}(ab)$.
  - ***right hand rule***: fingers curl from $\vec{a}$ to $\vec{b}$, thumb points along $\hat{n}$ perpendicular to $\vec{a}$ and $\vec{b}$.
  - Area of parallelogram $\vec{a},\vec{b}$ is $\left|\vec{a}\times\vec{b}\right| = a_\perp b$ (base × height), and $\hat{n}$ is normal to the area.
- Geometry: cross product is ***product of perpendicular projected lengths*** $|\vec{a}\times\vec{b}| = a_\perp b = ab_\perp$.
  - Either $a_\perp = a\sin\gamma$ ($\vec{a}$ against $\vec{b}$) or $b_\perp = b\sin\gamma$ ($\vec{b}$ onto $\vec{a}$), but not both against some $\vec{n}$
  - The projected product $a_\perp b$ is not the projected length $a_\perp$ unless $\vec{b} = \hat{b}$ is a unit vector.
  - ***projection***: $P_\perp\vec{v} \equiv \vec{v}_\perp = \hat{n}v_\perp = -\hat{n}\times(\hat{n}\times\vec{v})$ thus $P_\perp = -\hat{n}\times(\hat{n}\times$ is an ***operator*** $P_\perp : \vec{v} \mapsto \vec{v}_\perp$.
  - ***rotation***: after projecting, $\hat{n}\times\vec{v}$ rotates $\vec{v}$ $90°CCW$ about $\hat{n}$. Thus, it ***generates*** rotation.
- The ***triple product*** extends area to oriented volume using both the dot and cross product.
  - $V = \left(\vec{a}\times\vec{b}\right)\cdot\vec{c} = \vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = a_\parallel(b_\perp c) = a_\parallel A$ equals base area × perpendicular height.
  - It is totally antisymmetric and positive[negative] if $\vec{a},\vec{b},\vec{c}$ are ordered by the RHR[LHR].
  - The other 3 vector product: $\vec{a}\times\left(\vec{b}\times\vec{c}\right) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}\left(\vec{a}\cdot\vec{b}\right)$ reduces to vectors (BAC-CAB rule)
- History: Herman Grassmann introduced the exterior algebra of wedge products in 1844 to capture this symmetry in higher dimensions. A ***bivector*** $A = \vec{a}\wedge\vec{b}$ represents an area element in the plane (not its normal vector), and a trivector $V = \vec{a}\wedge\vec{b}\wedge\vec{c}$ represents a unit volume. William Hamilton generalized the complex numbers to quaterions $V = ijk$ to capture cross products (area, volume, and rotation) in 3-space. William Kingdon Clifford combined the symmetry of dot and cross products into the Clifford algebra $V = \vec{a}\vec{b}\vec{c}$ in 1973. These were all too complicated - Josiah Willard Gibbs and Oliver Heaviside reduced them to dot and cross products with the corresponding vector calculus, keeping the quaternion unit vectors i, j, k.
- The defining properties of the cross product relate the geometric → algebraic calculation:
  1) ***vector*** valued: (area). The cross product of polar vectors in an axial vector.
  2) ***antisymmetric*** (anticommutative): $\vec{a}\times\vec{b} = ab\sin\gamma = ba\sin(-\gamma) = -\vec{b}\times\vec{a}$.
  3) ***bilinear*** (distributive): $(\vec{a}_i\alpha_i)\times\left(\vec{b}_j\beta_j\right) = \alpha_i\left(\vec{a}_i\times\vec{b}_j\right)\beta_j = \alpha_i\vec{\epsilon}_{ij}\,\beta_j$, where $\vec{\epsilon}_{ij}$ is antisymmetric. this is a direct property of projections, head-to-tail addition, and similar triangles
  - Example - law of sines: area of $\triangle(abc)$ is $A = \frac{1}{2}\left|\vec{a}\times\vec{b}\right| = \frac{1}{2}ab\sin\gamma$, same for $\triangle(bca)$, etc.

Dividing by $\frac{1}{2}abc$, $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$. In the complex plane the area is, $A = \frac{1}{2}\mathrm{Im}\, a^* b$

- A right-handed ***orthonormal*** basis $\mathbb{e} = (\hat{x}, \hat{y}, \hat{z})$ follows the cylclic order $\hat{x} \to \hat{y} \to \hat{z} \to$ of the RHR.
  - $\hat{y} \times \hat{z} = \hat{x}$, $\hat{z} \times \hat{x} = \hat{y}$, $\hat{x} \times \hat{y} = \hat{z}$, and the opposites: $\hat{z} \times \hat{y} = -\hat{x}$, $\hat{x} \times \hat{z} = -\hat{y}$, $\hat{y} \times \hat{x} = -\hat{z}$.
  - $\vec{v} \times \vec{w} = \left(\hat{x}v_x + \hat{y}v_y + \hat{z}v_z\right) \cdot \left(\hat{x}w_x + \hat{y}w_y + \hat{z}w_z\right) = \hat{x}\left(v_y w_z - v_z w_y\right) + \hat{y}\left(v_y w_z - v_z w_y\right) + \hat{z}\left(v_y w_z - v_z w_y\right)$.
- Matrix notation - unlike the dot product, contractions are not as useful (but see below).
  We need a new matrix construct to characterize the antisymmetry of the cross product:
  - The ***determinant*** $|\mathbb{A}| = \begin{vmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{vmatrix} = \mathrm{ad} - \mathrm{bc}$ is the sum of all asymmetric signed products of exactly one element from each column, which also apear exactly once in each row.
    $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - afh - bdi - ceg$ (diagonals) for 3×3 matrices; $|a| = a$ for 1×1.
  - Comparing terms with above, you get the above matrix notation $\vec{a} \times \vec{b} = \left|\mathbb{e}^T \mathbb{a}\, \mathbb{b}\right| = \left|(\mathbb{e}^T \mathbb{a}\, \mathbb{b})^{\mathrm{T}}\right|$.
  - The triple product $\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = |\mathbb{a}\mathbb{b}\mathbb{c}|$ is also a determinant with $\mathbb{a}$ instead of $\mathbb{e}$ on top row.
    This gives the determinant its physical interpretation as the 'volume expansion' of matrix.
  - Properties of the determinant:
    1) Multiplicave: $|\mathbb{A}\mathbb{B}| = |\mathbb{A}||\mathbb{B}|$ consistent with definition of expansion factor.
    2) Linear in each row or column; thus $|\alpha\mathbb{A}| = \alpha^n |A|$, consistent with definition of volume.
    3) Antisymmetric in the exchange of any two rows or columns, which reverses orientation.
    4) $|\mathbb{A}| = 0$ if any two rows or columns are linearly dependent, so that the volume is zero.
    5) Symmetric: $|\mathbb{A}| = \left|\mathbb{A}^{\mathrm{T}}\right|$, since it follows the same pattern for rows and columns.
  - Higher $n \times n$ determinants have $n!$ terms, more than the $2n$ diagonals of the $2d$, $3d$ cases.
    We can calculate these with the ***cofactor expansion***, the contraction of elements A_{ij} in one row or column with their cofactors $C_{ij} = (-1)^{i+j}|\mathbb{A}_{ij}|$, the alternating subdeterminants of the matrix $\mathbb{A}_{ij}$, which is formed by crossing out the ith row and jth column of $\mathbb{A}$.
    $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$, but note that $d\begin{vmatrix} e & f \\ h & i \end{vmatrix} - e\begin{vmatrix} d & f \\ g & i \end{vmatrix} + f\begin{vmatrix} d & e \\ g & h \end{vmatrix} = 0$.
    This recursively preserves the definition of a completely antisymmetric product of elements
  - ***adjugate*** : $\mathrm{adj}(\mathbb{A}) = \mathbb{C}^T$ is the transpose of the ***cofactor matrix*** of alternating
    The adjugate has been transposed to contract with the cofactors in the expansion above.
    Thus, $\mathbb{A}\,\mathrm{adj}(\mathbb{A}) = |\mathbb{A}|\,\mathbb{I}$, or $\mathbb{A}^{-1} = \mathrm{adj}(\mathbb{A})\,/|\mathbb{A}|$; for example, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}/(ad - bc)$.
  - ***Cramer's rule***: for $\mathbb{A}\mathbb{x} = \mathbb{y}$, each component of the solution $\mathbb{y} = \mathbb{A}^{-1}\mathbb{x}$, involves a cofactor expansion of $\mathbb{x}$ with one row of $\mathrm{adj}(\mathbb{A})$, for the determinant of $\mathbb{A}^i$ which has $\mathbb{y}$ substituted into the ith column of $\mathbb{A}$. Thus: $y_i = \left|\mathbb{A}^i\right|/|\mathbb{A}|$; for example $y_1 = \frac{\begin{vmatrix} x_1 & b \\ x_2 & d \end{vmatrix}}{ad-bc}$ and $y_2 = \frac{\begin{vmatrix} a & x_1 \\ c & x_2 \end{vmatrix}}{ad-bc}$.
- Index notation: normal contraction pattern is useful for reducing formulas of components

- ***Levi-Civita tensor*** : $\vec{\mathcal{E}} = \hat{e}^{\mathrm{T}} \times \hat{e} = \begin{pmatrix} \vec{0} & \hat{x} & -\hat{y} \\ -\hat{x} & \vec{0} & \hat{z} \\ \hat{y} & -\hat{z} & \vec{0} \end{pmatrix} \sim \vec{\varepsilon}_{ij} = \hat{e}_i \times \hat{e}_j = \hat{e}_k \varepsilon_{ijk}$ (analog of the metric).
- This is a rank-3 tensor since it has rows, columns, and vector entries (three indices ε_{ijk}).
- Unlike the metric, this tensor (matrix) is rarely used with a non-orthonormal basis.
- Completely antisymmetric components: $\varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{ikj} = -\varepsilon_{kji}$, and $\varepsilon_{123} = 1$ by the RHR. Thus $\varepsilon_{iik} = \varepsilon_{iji} = \varepsilon_{ijj} = 0$ and $\mathcal{E}$ is cyclic $\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$ since $\hat{x} \times \hat{y} = \hat{z}$, all orthogonal by RHR.
- Not as useful for matrix manipulation as the metric because instead of $\mathbb{I}$ multiplying away, $\mathcal{E}$ is a cube of components; only matrix $\vec{\mathcal{E}}$ of vectors has natural matrix multiplication:

  $\vec{v} \times \vec{w} = (\hat{e}\mathbb{v}) \times (\hat{e}\mathbb{w}) = (\hat{e}\mathbb{v})^T \times (\hat{e}\mathbb{w}) = \mathbb{v}^{\mathrm{T}}(\hat{e}^T \times \hat{e})\, \mathbb{w} = \mathbb{v}^{\mathrm{T}}\, \vec{\mathcal{E}}\, \mathbb{w} = \hat{x} \ldots + \hat{y} \ldots + \hat{z} \ldots$

  However index notation $\vec{v} \times \vec{w} = \hat{e}_i v_i \times \hat{e}_j w_j = \hat{e}_i \times \hat{e}_j\, v_i w_j = \hat{e}_k \varepsilon_{ijk} v_i w_j$ is efficent to maniuplate, with three indices representing cubic matrices, and repeated indices specifing contractions.
- Similarly, $|\mathbb{A}| = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k} = \varepsilon_{ijk} A_{i1} A_{j2} A_{k3} = |\mathbb{A}^{\mathrm{T}}|$ explicitly describes the determinant as the sum of antisymmetric products of only one element in each row and column.
- Reduction of two cross products: $\varepsilon_{ijk}\varepsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} \equiv \delta^{ij}_{mn}$, the antisymmetric Kronecker delta, which equals ± 1 if i,j are an antisymmetric $\pm$ permuation of m,n.

  Example: the BAC-CAB rule: $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$ reduces to simple vectors.

  The kth component is $\left(\vec{a} \times (\vec{b} \times \vec{c})\right)_k = \varepsilon_{ijk} a_i (\vec{b} \times \vec{c})_j = \varepsilon_{ijk} a_i (\varepsilon_{jmn} b_m c_n) = (\varepsilon_{kij}\varepsilon_{jmn}) a_i b_m c_n$

  $= (\delta_{km}\delta_{in} - \delta_{kn}\delta_{im}) a_i b_m c_n = b_k a_i c_i - c_k a_i b_i = \left(\vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})\right)_k$, equal to the RHS.

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