L07 Operators

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- We have discussed three multilinear structures: $\cdot: V \times V \to \mathbb{R}$, $\times: V \times V \to V$, and $\times \cdot: V \times V \times V$.
  We also analyzed the projections: $\hat{n}\hat{n}\cdot: \mathrm{V} \to \mathrm{V}_{\parallel}$ and $-\hat{n} \times \hat{n} \times: V \to V_{\perp}$, which are linear operators.
  In fact, the dot, cross, tripple products are linear because their orthogonal projections are.
  - Definition: a ***map*** is a function $M: V_1 \to V_2: v_1 \mapsto v_2 = M(v_1)$ between spaces (ie vectors).
    It is usually drawn as arrows between gridded spaces, because of its higher dimensionality.
  - The ***composition*** of maps $M: V_1 \to V_2$ and $L: V_2 \to V_3$ is $L \circ M: V_1 \to V_3: v_1 \mapsto v_3 = L(M(v_1))$.
    Maps compose from right to left, since they act on the right (the rightmost acts first).
  - The ***identity*** is the map $I: V_1 \to V_2: v \to I(v) = v$. It acts like 1: $IM = M = MI$.
  - The ***inverse*** map: $M^{-1}: V_2 \to V_1: v_2 \mapsto v_1 = M^{-1}(v_2)$ where $M(v_1) = v(2)$. Thus $M^{-1}M = I = MM^{-1}$.
  - It ***linear*** if it respects linear combinations: $M(\vec{a}_i \alpha_i) = M(\vec{a}_i)\alpha_i$. You can perform the linear combination before or after mapping the the other space. This makes it easy to use. because a linear operator is completely defined by its action of a basis, so it has a similar geometrical structure as vectors themselves: if $\vec{e}_1, \vec{e}_2$ is a basis of V_1, then $M(\vec{e}_1)$, $M(\vec{e}_2)$ is a natural basis of V_2 and both have the 'eggcrate' grid structure.
  - The adjoint map $M^T: V_2 \to V_1$ respects the inner product: $\vec{v} \cdot M(\vec{w}) = M^T(\vec{v}) \cdot \vec{w}$.
    Note that the $\square^T$ notation is connected to the dot product, as promised.
  - An ***operator*** is a map (function) $M: V \to V$ on a vector space (it operates on vectors), and thus operators can be composed with themselves. Example: $P^2 = P$.
  - We will see that all linear operators can be decomposed into stretches and rotations, just like complex numbers.
- Components: linear maps (and operators) lend themselves to matrices, just like dot/cross. They contain the same contraction pattern we have seen many times already.
  - The components of a map form a matrix with elements from the action on basis vectors.
  - The elements M_{ij} of $\mathbb{M} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \end{pmatrix}$ are indexed by row, col (the same pattern).
  - The dimension of $M: V_1 \to V_2$, annotated ${}_m\mathbb{M}_{\mathrm{n}}$, where m, n are the number of $rows, cols$ (same again) represents the dimensions of $V_1 \sim \mathbb{R}^n$ and $V_2 \sim \mathbb{R}^m$ (note the backward order).
  - Composition of functions is represented by matrix multiplication (all possible contractions). The dimensions must align for the contraction: ${}_l\mathbb{K}_n = {}_l\mathbb{L}_m\, {}_m\mathbb{M}_{\mathrm{n}} = {}_l\mathbb{L}_m\mathbb{M}_n$.
  - The identity matrix $\mathbb{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ represents the identity operator I; thus, $\mathbb{IM} = \mathbb{M} = \mathbb{MI}$.
  - The determinant $|\mathbb{M}|$ represents the volume expansion of M by the triple product $|\vec{a}\vec{b}\vec{c}|$.

- The trace $\mathrm{Tr}\,\mathbb{M}$ represents the perimeter of M acting on the unit volume $(\hat{x}\hat{y}\hat{z})$.
- The inverse $\mathbb{M}^{-1} = \mathrm{adj}\,\mathbb{M}/|\mathbb{M}|$ is the matrix of M^{-1} (see L06).
- The transpose $\mathbb{M}^{\mathrm{T}} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \\ M_{31} & M_{32} \end{pmatrix}$ is the matrix of the adjoint M^T.
- Matrices can be ***active*** (from one space to another, or moving things around) or ***passive*** (components of the identity in two basis) for component transformations.

- Index notation
  - Reminder: components are indexed by x, y, z or equivalently $1,2,3$.
    indices i, j, k are variables that each represent either x, y, or z.
    - Each index should be repeated exactly twice (any exceptions should be explained)
    - An index repeated on both sides of the equation represents multiple equations
    - An index repeated on the same side of the equation is implicitly summed over
  - We already saw contraction (dot) and antisymmetric product (cross).
  - Matrix multiplication $K = LM$ is just many contractions $K_{ik} = \sum_{j=1}^{3} L_{ij}M_{jk} = L_{ij}M_{jk}$.
    Note: two indices j must be next-door neighbors to represent matrix multiplication.
    This proves that matrix multiplication is noncommutative: $LM \neq ML$ in general.
  - Multiple products: $(KLM)_{il} = K_{ij}L_{jk}M_{kl}$. From this we see matrix multiplication is associative.
  - Transpose: $\left(M^T\right)_{ij} = M_{ji}$. From this, we see that $(LM)^T = M^T L^T$.

- Components follow from linearity; and transform according to vector transformations.
  - Matrix elements: $\vec{M}(\vec{v}) = \vec{M}(\vec{\mathbb{e}}\mathbb{v}) = \vec{M}(\vec{\mathbb{e}})\mathbb{v} = \vec{\mathbb{e}}\mathbb{M}\mathbb{v}$ or $\vec{M}(\vec{v}) = \vec{M}\left(\vec{\mathbb{e}}_{\mathrm{j}}v_j\right) = \vec{M}\left(\vec{\mathbb{e}}_{\mathrm{j}}\right)v_j = \vec{M_j}v_j = \vec{\mathbb{e}}_{\mathrm{i}}M_{ij}v_j$.
    Thus, the jth column of $\mathbb{M}$ has the components of $\vec{M}\left(\vec{\mathbb{e}}_{\mathrm{j}}\right)$; there are two bases: rows,cols.
    If $\vec{w} = \vec{M}(\vec{v})$, then $\mathbb{w} = \mathbb{M}\mathbb{v}$. In an orthonormal basis where $\vec{w} = \hat{\mathbb{e}}\mathbb{w}$, then $\mathbb{M} = \hat{\mathbb{e}}^{\mathrm{T}} \cdot \vec{M}(\hat{\mathbb{e}})$.
  - Vector transformation: if $\mathbb{v}' = \mathbb{R}\mathbb{v}$ then $\vec{v} = \vec{\mathbb{e}}'\mathbb{v}' = \vec{\mathbb{e}}'(\mathbb{R}\mathbb{v}) = (\vec{\mathbb{e}}'\mathbb{R})\mathbb{v} = \vec{\mathbb{e}}\mathbb{v}$, so $\vec{\mathbb{e}}' = \vec{\mathbb{e}}\mathbb{R}^{-1}$
  - ***similarity*** transform: $\mathbb{w}' = \mathbb{R}(\mathbb{w} = \mathbb{M}\left(\mathbb{v} = \mathbb{R}^{-1}\mathbb{v}'\right) = \mathbb{R}\mathbb{M}\mathbb{R}^{-1}\mathbb{v}' = \mathbb{M}\mathbb{v}'$, so $\mathbb{M}' = \mathbb{R}\mathbb{M}\mathbb{R}^{-1}$
  - ***congruency*** transform: $\vec{v} \cdot \vec{\mathbb{w}} = \mathbb{v}'^{\mathrm{T}}\mathbb{g}'\mathbb{w}' = (\mathbb{R}\mathbb{v})^{\mathrm{T}}\mathbb{g}'(\mathbb{R}\mathbb{w}) = \mathbb{v}^{\mathrm{T}}(\mathbb{R}^{\mathrm{T}}\mathbb{g}'\mathbb{R})\mathbb{w}$, so $\mathbb{g}' = \mathbb{R}^{\mathrm{T}^{-1}}\mathbb{g}\mathbb{R}^{-1}$.
  - Orthogonal transformation: $\mathbb{v} = \mathbb{R}^{-1}\mathbb{v}' = \mathbb{R}^{\mathrm{T}}\mathbb{v}'$ so $\mathbb{M}' = \mathbb{R}\mathbb{M}\mathbb{R}^{\mathrm{T}}$ and $\mathbb{g}' = \mathbb{R}\mathbb{g}\mathbb{R}^T$ look the same.
    The basis transformation appears the same with $\vec{\mathbb{e}} \to \vec{\mathbb{e}}^{\mathrm{T}}$: so $\vec{\mathbb{e}}'^T = \left(\vec{\mathbb{e}}\mathbb{R}^{-1}\right)^T = \left(\vec{\mathbb{e}}\mathbb{R}^T\right)^T = \mathbb{R}\vec{\mathbb{e}}^T$.