L08 Rotations

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- The set of linear operators is restricted to $n \times n$ matrices, composed by multiplication. An operator M is represented by a grid of parallel (nonorthogonal) lines formed from linear combinations of the columns of $\mathbb{M}$ which are the components of $\vec{M}(\vec{e}_i)$.
  - You can visualize this as shearing a grid, keeping parallel lines parallel.
  - In general, any operator $M = RS$ is the composition of a ***rotation*** R and a ***stretch*** S, (polar decomposition theorem), in analogy with the complex polar representation $z = \rho e^{i\phi}$.
  - We will treat rotations first, since they are needed to describe stretches.
- Derivation of 2d rotation matrix from $e^{i\phi}$ in the complex plane
  - $x' + iy' = z' = e^{i\phi}z = \left(c_\phi + is_\phi\right)(x + iy) = \left(c_\phi + is_\phi\right)x + \left(-s_\phi + ic_\phi\right)y = \left(c_\phi x - s_\phi y\right) + i\left(s_\phi x + c_\phi y\right)$.
  - The matrix equation for this rotation $\vec{v}' = \vec{R}(\vec{v})$ is $\mathbb{v}' = \mathbb{R}\mathbb{v}$ or $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} c_\phi & -s_\phi \\ s_\phi & c_\phi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.
  - The rows of $\mathbb{R}$ come from real, imag parts of z': $x' = c_\phi x - s_\phi y$ and $y' = s_\phi x + c_\phi y$.
  - The cols of $\mathbb{R}$ are the unit vectors $\hat{x}, \hat{y}$ rotated: $\hat{x}' = \hat{x}c_\phi + \hat{y}s_\phi$ and $\hat{x}' = -\hat{x}s_\phi + \hat{y}c_\phi$.
  - Note: the row equations are contractions, while the col equations are linear combinations.
- Active versus Passive rotations
  - An ***active*** rotation physically rotates vectors including unit vectors $\hat{\mathbb{x}} \sim \mathbb{e}_1$ and $\hat{\mathbb{y}} \sim \mathbb{e}_2$ into new vectors, like $\mathbb{e}_1'$ and $\mathbb{e}_2'$, with different components in the same basis.
    - The new components are $\mathbb{e}_1' = \mathbb{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbb{e}_2' = \mathbb{R}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which augments to $\left(\mathbb{e}_1'\mathbb{e}_2'\right) = \mathbb{R}\mathbb{I} = \mathbb{R}$.
  - A ***passive*** rotation or basis transformation rotates the basis into a linear combination of the old one: $\left(\hat{x}' \quad \hat{y}'\right) = \left(\hat{x} \quad \hat{y}\right)\begin{pmatrix} c_\phi & -s_\phi \\ s_\phi & c_\phi \end{pmatrix}$ or $\hat{\mathbb{e}}' = \hat{\mathbb{e}}\mathbb{R}$. Note: $\hat{\mathbb{e}}, \hat{\mathbb{e}}'$ are vectors, not components.
    - Physical vectors $\vec{v} = \hat{\mathbb{e}}\mathbb{v}$ are not rotated, they just change components in a new basis: $\vec{v} = \hat{\mathbb{e}}'\mathbb{v}' = \hat{\mathbb{e}}\mathbb{R}\mathbb{v}' = \hat{\mathbb{e}}\mathbb{v}$, and thus $\mathbb{v} = \mathbb{R}\mathbb{v}'$ or $\mathbb{v}' = \mathbb{R}^{-1}\mathbb{v}$, which is in the opposition direction.
    - If you tip your head (reference) to the left, the vector looks like it rotates to the right. Thus, normal vectors components are called ***contravariant***–they vary against the basis.
    - A foolproof method to perform passive transformations of components is to write $\vec{v} = \vec{b}_1 v_1 + \vec{b}_2 v_2 + \cdots$, substitute $\vec{b}_1 = \vec{b}_1' \ldots + \vec{b}_2' \ldots$, etc., and collect components of $\vec{b}_1'$, etc.
- Properties of rotations
  - Orthogonal transformation: $\mathbb{R}^{\mathrm{T}}\mathbb{R} = \mathbb{I}$

    $\mathbb{R}^{\mathrm{T}}\mathbb{R} = \begin{pmatrix} c_\phi & s_\phi \\ -s_\phi & c_\phi \end{pmatrix}\begin{pmatrix} c_\phi & -s_\phi \\ s_\phi & c_\phi \end{pmatrix} = \begin{pmatrix} c_\phi c_\phi + s_\phi s_\phi & -c_\phi s_\phi + s_\phi c_\phi \\ -s_\phi c_\phi + c_\phi s_\phi & s_\phi s_\phi + c_\phi c_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}$.
    - This is true in general, as seen by treating ^{T} as the dot product of columns

$$\mathbb{R}^{\mathrm{T}}\mathbb{R} = \begin{pmatrix} \mathbb{e}_1'^T \\ \mathbb{e}_2'^T \end{pmatrix} (\mathbb{e}_1' \quad \mathbb{e}_2') = \begin{pmatrix} \hat{e}_1' \\ \hat{e}_2' \end{pmatrix} \cdot (\hat{e}_1' \quad \hat{e}_2') = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}.$$

- Rotation (orthogonal transformation) preserves shape (lengths/angles or dot product).
- The most general orthogonal transformation preserves the metric: $\mathbb{g}' = \mathbb{R}^T \mathbb{g} \mathbb{R} = \mathbb{g}$

- Composition of rotations in the xy-plane: (in general, the 'angle' is an axial vector

$$\mathbb{R}_\phi \mathbb{R}_\psi = \begin{pmatrix} c_\phi & -s_\phi \\ s_\phi & c_\phi \end{pmatrix} \begin{pmatrix} c_\psi & -s_\psi \\ s_\psi & c_\psi \end{pmatrix} = \begin{pmatrix} c_\phi c_\psi - s_\phi s_\psi & -c_\phi s_\psi - s_\phi c_\psi \\ c_\phi s_\psi + s_\phi c_\psi & s_\phi s_\psi + c_\phi c_\psi \end{pmatrix} = \begin{pmatrix} c_{\phi+\psi} & -s_{\phi+\psi} \\ s_{\phi+\psi} & c_{\phi+\psi} \end{pmatrix} = \mathbb{R}_{\phi+\psi}.R$$

- Rotations in the same plane commute $\mathbb{R}_\phi \mathbb{R}_\psi = \mathbb{R}_\psi \mathbb{R}_\phi$. Even if they don't commute,
- the product $\mathbb{R} = \mathbb{R}_2 \mathbb{R}_1$ is a rotation: $\mathbb{R}^T \mathbb{R} = (\mathbb{R}_2 \mathbb{R}_1)^T (\mathbb{R}_2 \mathbb{R}_1) = \mathbb{R}_1^T (\mathbb{R}_2^T \mathbb{R}_2) \mathbb{R}_1 = \mathbb{R}_1 \mathbb{R}_1 = \mathbb{I}$.

- Inverse rotation: $\mathbb{R}^{-1} = \mathbb{R}^{\mathrm{T}}$, from $(\mathbb{R}^{\mathrm{T}}\mathbb{R} = \mathbb{I})\mathbb{R}^{-1}$

$$\mathbb{R}_\phi^{-1} = \mathbb{R}_{-\phi} = \begin{pmatrix} c_{-\phi} & -s_{-\phi} \\ s_{-\phi} & c_{-\phi} \end{pmatrix} = \begin{pmatrix} c_\phi & s_\phi \\ -s_\phi & c_\phi \end{pmatrix} = \mathbb{R}^{\mathrm{T}}.$$

- This makes it trivial to invert a rotation, which is needed to find components
- This also simplifies similarity and congruence transforms as seen in L07.

- Rotations form a group under composition (matrix multiplication).

- Symmetry of rotation matrices
  - Like the complex rotation $e^{-i\phi}$, a rotation operator is $R = e^A$, where A is antisymmetric.
  - Any operator (matrix) has the decomposition $M = S + A$, where $S^T = S$ and $A^T = -A$, as seen by taking the transpose $M^T = S^T + A^T$ and solving the two equations for S, A.
  - Orthogonality: $R^T R = e^{A^T} e^A = e^{-A} e^A = e^{A-A} = e^0 = I$.
  - Inverse: $R^{-1} = (e^A)^{-1} = e^{-A} = e^{A^T} = (e^A)^T = R^T$.
- 3d rotation matrices
  - Along the z-axis, x, y rotate as above and the z component stays the same:

$$\mathbb{R}_{\hat{z}\phi} = \begin{pmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{R}_\phi \oplus \mathbb{I}$$. Similarly, $$\mathbb{R}_{\hat{y}\theta} = \begin{pmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{pmatrix}$$, with cyclic ordering.

  - Like general matrices, 3d rotations do not commute unless along the same axis.
  - Any 3d rotation can be constructed from three rotations (Euler/Tate angles)

$$\mathbb{R}_{\psi,\theta,\phi} = \mathbb{R}_{\hat{z}\phi} \mathbb{R}_{\hat{y}\theta} \mathbb{R}_{\hat{z}\psi} = \begin{pmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{pmatrix} \begin{pmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

  - Any rotation can be identified with an axial vector $\vec{v} = \hat{n}\phi$, where $\hat{n}$ is the axis of rotation. Rodrigues' formula: $R = e^{\vec{\mathcal{E}} \cdot \vec{v}} = I c_\phi + \vec{\mathcal{E}} \cdot \hat{n} s_\phi + (1 - c_\phi)\hat{n}\hat{n} \cdot$ Last term projects R in the plane.