L09 Stretches

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- If an operator simply stretches a vector, then we can replace its matrix by the stretch factor.
  - Such vectors are called ***eigenvectors*** $\vec{v}_i$ and the corresponding stretch factors ***eigenvalues*** λ_i.
  - The n eigenvalue equations are $M\vec{v}_i = \vec{v}_i\lambda_i$, which can be augmented to $\mathbb{M}\mathbb{V} = \mathbb{V}\mathbb{D}$, where the columns of $\mathbb{V}$ are components of $\vec{v}_i$, and $\mathbb{D}$ is a diagonal matrix of λ_i's. Thus the process of finding eigen{values,vector} is called ***diagonalization***. The matrix $\mathbb{V}$ is the ***eigenbasis***, and $\mathbb{D}$ is the ***spectrum*** of $\mathbb{M}$ in reference to QM.
  - Most $n \times n$ matrices have n independent eigenvectors and n corresponding eigenvalues.
    - All have n eigenvalues, but some may be repeated (algebraic multiplicity).
    - A repeated eigenvalue may have fewer eigenvectors (geometric multiplicity); in this case, the matrix has less than n eigenvectors and is called ***defective***.
    - Singular matrices are not necessarily defective, they just have a 0 eigenvalue.
- Example: $\mathbb{M} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has 2 eigenvectors: $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} 3$ and $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
  Thus $\mathbb{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda_1 = 3$ and $\mathbb{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\lambda_2 = 1$, combined into $\mathbb{V} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\mathbb{D} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.
  The effect is easiest seen on the action of M on the unit circle.
- To calculate both the eigenvalues and eigenvectors, we must
  1) reduce 2 unknowns to 1: $\mathbb{M}\mathbb{v} = \mathbb{v}\lambda = \lambda\mathbb{I}\mathbb{v}$ or $(\mathbb{M} - \lambda\mathbb{I})\mathbb{v} = 0$ for nonzero $\mathbb{v}$, which implies $|\mathbb{M} - \lambda\mathbb{I}| = 0$. The determinant is an nth order polynomial in λ with n roots by the FTA.
     Example: $|\mathbb{M} - \lambda\mathbb{I}| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 \cdot 1 = 0$, so $\lambda = 2 \pm \sqrt{1}$ or $\lambda_1 = 3$, $\lambda_2 = 1$.
  2) For each λ_i solve the equation $(\mathbb{M} - \lambda_i\mathbb{I})\mathbb{v}_\mathrm{i} = 0$ for $\mathbb{v}$. It is degenerate by design.
     There is not a unique solution, but a whole line called the ***eigenspace*** of λ_i. If λ_i is degenerate, there is a whole plane (or higher dimension) of solutions, unless $\mathbb{M}$ is defective.
     Example: $(\mathbb{M} - \lambda_1\mathbb{I})\mathbb{v}_1 = \begin{pmatrix} 2-3 & 1 \\ 1 & 2-3 \end{pmatrix}\begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Both the top and bottom equations are the same: $v_{1x} = v_{1y}$ or $\mathbb{v}_1 \propto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and likewise, $v_{2x} = -v_{2y}$ or $\mathbb{v}_2 \propto \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. You often normalize $\mathbb{v}_\mathrm{i}$, which makes it unique up to an arbitrary sign in the nondegenerate case.
- Properties of stretches
  1) Diagonal matrices multiply elementwise $\mathbb{D}\mathbb{G} = \mathrm{diag}(d_1 g_1,\ d_2 g_2, \ldots)$ decoupling each dimension. Therefore, they also commute $\mathbb{D}\mathbb{G} = \mathbb{G}\mathbb{D}$. They represent stretches along $\hat{x}, \hat{y}, \hat{z}$.
  2) The ***eigendecomposition*** of $\mathbb{M} = \mathbb{V}\mathbb{D}\mathbb{V}^{-1} = \sum_\mathrm{i} P_i\lambda_i$ into outer products stretches along each individual projection. This similarity transform can also be thought of as transforming the eigenvectors to $\hat{x}, \hat{y}, \hat{z}$, stretching along these axes, and transforming back to $\vec{v}_1, \vec{v}_2, \vec{v}_3$.
     It simplifies $= \sum_\mathrm{k} f_k \mathbb{M}^\mathrm{k}/k! = f_0\mathbb{I} + \mathrm{f}_1\mathbb{M} + f_2\,\mathbb{M}^2/2 = f_0\mathbb{V}\mathbb{V}^{-1} + \mathrm{f}_1\mathbb{V}\mathbb{D}\mathbb{V}^{-1} + f_2\,\mathbb{V}\mathbb{D}\mathbb{V}^{-1}\,\mathbb{V}\mathbb{D}\mathbb{V}^{-1}/2$
     $= \mathbb{V}\left(f_0\mathbb{I} + \mathrm{f}_1\mathbb{D} + f_2\,\mathbb{D}^2/2 + \cdots\right)\mathbb{V}^{-1} = \mathbb{V}\,f(\mathbb{D})\,\mathbb{V}^{-1} = \mathbb{V}\begin{pmatrix} f(\lambda_1) & 0 \\ 0 & f(\lambda_2) \end{pmatrix}\mathbb{V}^{-1}$, for example, $f(\mathbb{M}) = \mathrm{e}^{\mathbb{M}}$.

3) A ***symmetric*** operator $S^T = S$ is nondefective and has orthogonal eigenspaces, so $\mathbb{V}^{\mathrm{T}}\mathbb{V} = \mathbb{I}$, and real eigenvalues. In that case, $\mathbb{M} = \mathbb{V}\mathbb{D}\mathbb{V}^{\mathrm{T}}$ and $\mathbb{D} = \mathbb{V}^{\mathrm{T}}\mathbb{M}\mathbb{V}$, which simplifies inverting $\mathbb{V}$. The projectors simplify to $P_i = \mathbb{v}_i\mathbb{v}_i^T$, an orthogonal projection along $\mathbb{v}_i$.

4) A ***normal*** operator $N = H + iK$, where $N^T N = NN^T$, has commuting symmetric $H^\dagger = H$ and antisymmetric $(iK)^\dagger = -iK$ parts, which share the same eigenbasis $\mathbb{V}$. Thus N is nondefective and $\mathbb{V}$ is orthogonal $\mathbb{V}^{\mathrm{T}}\mathbb{V} = \mathbb{I}$, but the eigenvalues are complex in general. This includes:

   i. Symmetric matrices with real eigenvalues

   ii. Antisymmetric matrices with imaginary eigenvalues

   iii. Unitary [orthogonal] matrices with unit $e^{i\phi}$ eigenvalues [in $e^{\pm i\phi}$ pairs].

   iv. Orthogonal projections with eigenvalues of 1's and 0's.

   The Hermitian adjoint corresponds to taking the complex conjugate of the eigenvalues. They essentially act like n indepdent complex eigenvalues, per the normal matrix analogy.

- Polar Decomposition and Singular Value Decomposition (SVD)
  - Any operator M has a generator G, such that $M = e^G$.
    Decompose $G = A + T$ into its antisymmetric $A^T = -A$ and symmetric $T^T = T$ parts.
    If they commute, then $M = e^G = e^{A+T} = e^A e^T = RS$, where $R^T R = e^{A-A} = I$ and $S^T = S$.
    Even if the don't, in general, $M = RS$ where $R^T R = I$ and $S^T = S$. (polar decomposition)
  - Diagonalize $S = VWV^T$ and combine rotations $U = RV$ to obtain the SVD:
    $M = RS = RVWV^T = UWV^T$, where $U^T U = I$ and $V^T V = I$.
    This is a generalized eigendecomposition which is always orthogonal, and even works on general linear maps from one space to another (with rectangular matrices).
    The SVD and FFT (L12) are two of the most important numerical algorithms.