Formula Sheet

Thursday, April 17, 2025

- Algebra

$+$: $a+b=b+a \quad a+(b+c)=(a+b)+c$

$-$: $0+a=a \quad a+(-a)=0 \quad --a=a \quad -(a+b)=-a+-b \quad a-b\equiv a+(-b)$

$\times$: $ab=ba \quad a(bc)=(ab)c$

$\div$: $1a=a \quad a(a^-)=1 \quad (a^-)^-=a \quad (ab)^-=a^-b^- \quad a/b\equiv a(b^-) \quad (a/b)^-=b/a$

$+\times$: $a(b+c)=ab+ac \quad 0a=0 \quad -1\,a=-a \quad nx=x+x+\cdots+x$

^: $a^n=aa\ldots a \quad 0^c=0 \quad 1^c=1 \quad a^0=1 \quad a^1=a \quad a^{-1}=a^-=1/a$

$\times$ ^ $+$: $a^{b+c}=a^ba^c \quad a^{bc}=\left(a^b\right)^c \quad (ab)^c=a^cb^c \quad (a+b)^n=\sum_k\binom{n}{k}a^{n-k}b^k \quad \binom{n}{k}=\frac{n!}{k!(n-k)!}$

$\surd$: $\sqrt[b]{a^b}=\sqrt[b]{a}^{\,b}=a \quad \sqrt[b]{a}\equiv a^{1/b} \quad \sqrt{a}\equiv\sqrt[2]{a} \quad \sqrt{ab}=\sqrt{a}\sqrt{b} \quad \sqrt{\sqrt{a}}=\sqrt[4]{a}$

ln: $a=b^c\leftrightarrow c=\log_b a=\ln a/\ln b \quad \ln ab=\ln a+\ln b \quad \ln b^c=c\ln b \quad \ln e=1$

i: $i^2=-1 \quad z=x+iy=\rho e^{i\phi} \quad z^*=x-iy=\rho e^{-i\phi} \quad |z|^2\equiv z^*z=x^2+y^2$

$(a-b)(a+b)=a^2-b^2 \quad ax^2+bx+c=a(x-b/2a)^2+(c-b^2/4a)$

FTA: $a_0+a_1x+a_2x^2+\cdots a_nx^n=a_n(x-x_1)(x-x_2)\ldots(x-x_n) \quad x_{2k-1}=x_{2k}^*$ *if* $a_i\in\mathbb{R}$

- Trigonometry

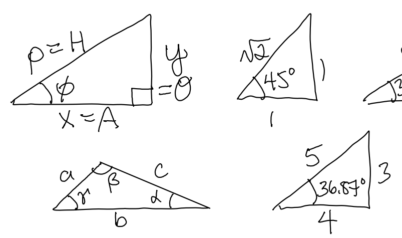
SOH CAH TOA

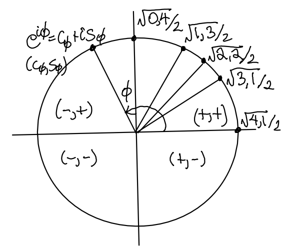
$x=\rho c_\phi \quad y=\rho s_\phi$

$x^2+y^2=\rho^2 \quad y/x=t_\phi$

$c^2=a^2+b^2-2abc_\gamma$

$a/s_\alpha=b/s_\beta=c/s_\gamma$





$e^{i\phi}=c_\phi+is_\phi\equiv cis_\phi \quad c_\phi=\frac{1}{2}(e^{i\phi}+e^{-i\phi}) \quad s_\phi=\frac{1}{2i}(e^{i\phi}-e^{-i\phi}) \quad i^2=-1$

$c_{\phi\pm\psi}=c_\phi c_\psi\mp s_\phi s_\psi \quad c_\phi^2=\frac{1}{2}\left(1+c_{2\phi}\right) \quad c_{2\phi}=c_\phi^2-s_\phi^2 \quad c_\phi^2+s_\phi^2=1 \quad \tan^2\phi+1=\sec^2\phi$

$s_{\phi\pm\psi}=s_\phi c_\psi\pm c_\phi s_\psi \quad s_\phi^2=\frac{1}{2}\left(1-c_{2\phi}\right) \quad s_{2\phi}=2c_\phi s_\phi \quad \left(c_\phi\pm s_\phi\right)^2=1\pm s_{2\phi}$

$e^{h\alpha}=\bar{c}_\alpha+h\bar{s}_\alpha\equiv chs_\alpha \quad \bar{c}_\alpha=\frac{1}{2}(e^{h\alpha}+e^{-h\alpha}) \quad \bar{s}_\alpha=\frac{1}{2h}(e^{h\alpha}-e^{-h\alpha}) \quad h^2=1 \quad h=\pm1$

$\bar{c}_{\alpha\pm\beta}=\bar{c}_\alpha\bar{c}_\beta\pm\bar{s}_\alpha\bar{s}_\beta \quad \bar{c}_\alpha^2=\frac{1}{2}(\bar{c}_{2\alpha}+1) \quad \bar{c}_{2\alpha}=\bar{c}_\alpha^2+\bar{s}_\alpha^2 \quad \bar{c}_\alpha^2-\bar{s}_\alpha^2=1 \quad \tanh^2\alpha+1=\operatorname{sech}^2\alpha$

$\bar{s}_{\alpha\pm\beta}=\bar{s}_\alpha\bar{c}_\beta\pm\bar{c}_\alpha\bar{s}_\beta \quad \bar{s}_\alpha^2=\frac{1}{2}(\bar{c}_{2\alpha}-1) \quad \bar{s}_{2\alpha}=2\bar{c}_\alpha\bar{s}_\alpha \quad (\bar{c}_\alpha\pm\bar{s}_\alpha)^2=\bar{c}_{2\alpha}\pm\bar{s}_{2\alpha}$

$c_\phi=\bar{c}_{i\phi} \quad \bar{c}_\alpha=c_{-i\alpha} \quad s_\phi=-i\bar{s}_{i\phi} \quad \bar{s}_\alpha=is_{i\alpha} \quad \alpha=i\phi$

- Linear Algebra

$\vec{\boldsymbol{A}}=\hat{\boldsymbol{x}}A_x+\hat{\boldsymbol{y}}A_y+\hat{\boldsymbol{z}}A_z \quad \vec{\boldsymbol{u}}+\vec{\boldsymbol{v}}=\vec{\boldsymbol{v}}+\vec{\boldsymbol{u}} \quad \alpha(\beta\vec{\boldsymbol{u}})=(\alpha\beta)\vec{\boldsymbol{u}} \quad \alpha(\vec{\boldsymbol{u}}+\vec{\boldsymbol{v}})=\alpha\vec{\boldsymbol{u}}+\alpha\vec{\boldsymbol{v}}$

$\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}}=\vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{A}}=A_xB_x+A_yB_y+A_zB_z \quad (\alpha_i\vec{\boldsymbol{u}}_i)\cdot\left(\vec{\boldsymbol{v}}_j\beta_j\right)=\alpha_i\left(\vec{\boldsymbol{u}}_i\cdot\vec{\boldsymbol{v}}_j\right)\beta_j$ (same for $\times$)

$\vec{\boldsymbol{B}}\times\vec{\boldsymbol{C}}=-\vec{\boldsymbol{C}}\times\vec{\boldsymbol{B}}=\hat{\boldsymbol{x}}\left(B_yC_z-B_zC_y\right)+\hat{\boldsymbol{y}}(B_zC_x-B_xC_z)+\hat{\boldsymbol{z}}\left(B_xC_y-B_yC_x\right)=\begin{vmatrix}\hat{\boldsymbol{x}}&\hat{\boldsymbol{y}}&\hat{\boldsymbol{z}}\\B_x&B_y&B_z\\C_x&C_y&C_z\end{vmatrix}$

$\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}}\times\vec{\boldsymbol{C}}=\vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{C}}\times\vec{\boldsymbol{A}}=\vec{\boldsymbol{C}}\cdot\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}=\begin{vmatrix}A_x&A_y&A_z\\B_x&B_y&B_z\\C_x&C_y&C_z\end{vmatrix}$

$\vec{\boldsymbol{A}}\times\left(\vec{\boldsymbol{B}}\times\vec{\boldsymbol{C}}\right)=\vec{\boldsymbol{B}}\left(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{C}}\right)-\vec{\boldsymbol{C}}\left(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}}\right)$

$P_\parallel+P_\perp=\hat{n}\hat{n}\cdot\ -\hat{n}\times\hat{n}\times=I$

$\det A\equiv\begin{vmatrix}a&b\\c&d\end{vmatrix}=ad-bc \quad \operatorname{tr}A=A_{ii}=a+d \quad A^{-1}\equiv\begin{pmatrix}a&b\\c&d\end{pmatrix}^{-1}=\frac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$

$M=(\,M(\hat{x}),\ M(\hat{y}),\ M(\hat{z})\,) \quad M(\alpha_i\vec{v}_i)=\alpha_iM(\vec{v}_i)$

$R^TR=I \quad R^{-1}=R^T \quad |R|=1$ (orthogonal)

$S^T=S \quad S\vec{\boldsymbol{v}}_i=\vec{\boldsymbol{v}}_i\lambda_i \quad SV=VD \quad S=VDV^{-1} \quad V^TV=I$

$\langle f|g\rangle=\int w(x)dx\,f^*(x)g(x) \quad |g\rangle=g(x) \quad \langle f|=\int wdx\,f^*(x)\ldots$

$M=\frac{1}{w(x)}\left(\frac{d}{dx}p(x)\frac{d}{dx}+q(x)\right) \quad M^\dagger=M \ \Rightarrow\ \lambda_i\in\mathbb{R},\ \langle u_i|u_j\rangle=\delta_{ij}h_i^2$

- *Calculus*

  $\frac{d}{dx}x^\alpha = \alpha x^{\alpha-1}$ $\frac{d}{dx}e^x = e^x$ $\frac{d}{dx}\ln x = \frac{1}{x}$ $\frac{d}{d\phi}\sin\phi = \cos\phi$ $\frac{d}{d\phi}\cos\phi = -\sin\phi$ $\frac{d}{d\phi}\tan\phi = \sec^2\phi$

  $d(f+g) = df + dg$ $d(fg) = g\,df + f\,dg$ $d(f/g) = (g\,df - f\,dg)/g^2$ $d(f\circ g) = \frac{df}{dg}dg$

  $e^x = \sum x^k/_{k!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots$ $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$

  $e^{i\phi} = c_\phi + is_\phi$ $\cos\phi = 1 - \frac{1}{2}\phi^2 + \frac{1}{24}\phi^4 - \cdots$ $\sin\phi = \phi - \frac{1}{6}\phi^3 + \frac{1}{120}\phi^5 - \cdots$

  $(1+u)^\alpha = \sum\binom{\alpha}{k}u^k = 1 + \frac{\alpha}{1}u + \frac{\alpha(\alpha-1)}{1\cdot 2}u^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{1\cdot 2\cdot 3}u^3 + \cdots$

- *Vector Calculus*

  $d = dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z} + \cdots$ $\boldsymbol{\nabla} = (\partial_x, \partial_y, \partial_z) = \hat{\boldsymbol{x}}\partial_x + \hat{\boldsymbol{y}}\partial_y + \hat{\boldsymbol{z}}\partial_z$ *(Cartesian only)*

  $\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla}g + g\boldsymbol{\nabla}f$ $\boldsymbol{\nabla}f(g(\vec{\boldsymbol{r}}), h(\vec{\boldsymbol{r}}), \ldots) = f_{,g}\,\boldsymbol{\nabla}g + f_{,h}\,\boldsymbol{\nabla}h + \cdots$

  $\boldsymbol{\nabla}\times(f\vec{\boldsymbol{A}}) = f\boldsymbol{\nabla}\times\vec{\boldsymbol{A}} - \vec{\boldsymbol{A}}\times\boldsymbol{\nabla}f$ $\boldsymbol{\nabla}\times\vec{\boldsymbol{A}}(g(\vec{\boldsymbol{r}}), h(\vec{\boldsymbol{r}}), \ldots) = -\vec{\boldsymbol{A}}_{,g}\times\boldsymbol{\nabla}g - \vec{\boldsymbol{A}}_{,h}\times\boldsymbol{\nabla}h + \cdots$

  $\boldsymbol{\nabla}\cdot(f\vec{\boldsymbol{A}}) = f(\boldsymbol{\nabla}\cdot\vec{\boldsymbol{A}}) + \boldsymbol{\nabla}f\cdot\vec{\boldsymbol{A}}$ $\boldsymbol{\nabla}\cdot\vec{\boldsymbol{B}}(g(\vec{\boldsymbol{r}}), h(\vec{\boldsymbol{r}}), \ldots) = \vec{\boldsymbol{B}}_{,g}\cdot\boldsymbol{\nabla}g + \vec{\boldsymbol{B}}_{,h}\cdot\boldsymbol{\nabla}h + \cdots$ $\int_P \nabla f\cdot\overrightarrow{\boldsymbol{dl}} = \oint_{\partial P} f$

  $\boldsymbol{\nabla}\cdot(\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}) = \vec{\boldsymbol{B}}\cdot(\boldsymbol{\nabla}\times\vec{\boldsymbol{A}}) - \vec{\boldsymbol{A}}\cdot(\boldsymbol{\nabla}\times\vec{\boldsymbol{B}})$ $\int_S \nabla\times\vec{\boldsymbol{A}}\cdot\overrightarrow{\boldsymbol{da}} = \oint_{\partial S}\vec{\boldsymbol{A}}\cdot\overrightarrow{\boldsymbol{dl}}$

  $\boldsymbol{\nabla}(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}}) = \vec{\boldsymbol{A}}\times(\boldsymbol{\nabla}\times\vec{\boldsymbol{B}}) + \vec{\boldsymbol{B}}\times(\boldsymbol{\nabla}\times\vec{\boldsymbol{A}}) + (\vec{\boldsymbol{A}}\cdot\boldsymbol{\nabla})\vec{\boldsymbol{B}} + (\vec{\boldsymbol{B}}\cdot\boldsymbol{\nabla})\vec{\boldsymbol{A}}$

  $\boldsymbol{\nabla}\times(\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}) = (\vec{\boldsymbol{B}}\cdot\boldsymbol{\nabla})\vec{\boldsymbol{A}} - (\vec{\boldsymbol{A}}\cdot\boldsymbol{\nabla})\vec{\boldsymbol{B}} + \vec{\boldsymbol{A}}(\boldsymbol{\nabla}\cdot\vec{\boldsymbol{B}}) - \vec{\boldsymbol{B}}(\boldsymbol{\nabla}\cdot\vec{\boldsymbol{A}})$ $\int_V \nabla\cdot\vec{\boldsymbol{B}}\cdot d\tau = \oint_{\partial V}\vec{\boldsymbol{B}}\cdot\overrightarrow{\boldsymbol{da}}$

  $\boldsymbol{\nabla}\times\vec{\boldsymbol{A}} = \vec{\boldsymbol{0}} \Leftrightarrow \vec{\boldsymbol{A}} = -\boldsymbol{\nabla}f$ $\boldsymbol{\nabla}\cdot\vec{\boldsymbol{B}} = 0 \Leftrightarrow \vec{\boldsymbol{B}} = \boldsymbol{\nabla}\times\vec{\boldsymbol{A}}$ $\nabla^2 f = (\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})f$ $\nabla^2\vec{\boldsymbol{A}} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\vec{\boldsymbol{A}}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\vec{\boldsymbol{A}})$

  Coordinate transformations … *and inverse…* *summary…*

  $x = \rho c_\phi = rs_\theta c_\phi$ $z = rc_\theta$ $\rho^2 = x^2 + y^2$ $r^2 = \rho^2 + z^2$ $x + iy = \rho e^{i\phi}$

  $y = \rho s_\phi = rs_\theta s_\phi$ $\rho = rs_\theta$ $t_\phi = y/x$ $t_\theta = \rho/z$ $z + i\rho = re^{i\theta}$

  Unit vector transformations … *and inverse…* *summary…*

  $\hat{x}\rho = \hat{\rho}x - \hat{\phi}y$ $\hat{\rho}r = \hat{r}\rho - \hat{\theta}z$ $\hat{\rho}\rho = \hat{x}x + \hat{y}y$ $\hat{r}r = \hat{\rho}\rho + \hat{z}z$ $\hat{\rho} + i\hat{\phi} = (\hat{x} - i\hat{y})e^{i\phi}$

  $\hat{y}\rho = \hat{\rho}y + \hat{\phi}x$ $\hat{z}r = \hat{r}z + \hat{\theta}\rho$ $\hat{\phi}\rho = -\hat{x}y + \hat{y}x$ $\hat{\theta}r = \hat{\rho}z - \hat{z}\rho$ $\hat{r} + i\hat{\theta} = (\hat{z} - i\hat{\rho})e^{i\theta}$

  $\hat{x} = \hat{\rho}c_\phi - \hat{\phi}s_\phi$ $\hat{\rho} = \hat{r}s_\theta - \hat{\theta}c_\theta$ $\hat{\rho} = \hat{x}c_\phi + \hat{y}s_\phi$ $\hat{r} = \hat{\rho}s_\theta + \hat{z}c_\theta = \hat{x}s_\theta c_\phi + \hat{y}s_\theta s_\phi + \hat{z}c_\theta$

  $\hat{y} = \hat{\rho}s_\phi + \hat{\phi}c_\phi$ $\hat{z} = \hat{r}c_\theta + \hat{\theta}s_\theta$ $\hat{\phi} = -\hat{x}s_\phi + \hat{y}c_\phi$ $\hat{\theta} = \hat{\rho}c_\theta - \hat{z}s_\theta = \hat{x}c_\theta c_\phi + \hat{y}c_\theta s_\phi - \hat{z}s_\theta$

  Cartesian coordinates (x, y, z) $\vec{\boldsymbol{r}} = \hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y + \hat{\boldsymbol{z}}z$ $h_x = h_y = h_z = 1$

  $\overrightarrow{\boldsymbol{dl}} = d\vec{\boldsymbol{r}} = \hat{\boldsymbol{x}}dx + \hat{\boldsymbol{y}}dy + \hat{\boldsymbol{z}}dz$ $\overrightarrow{\boldsymbol{da}} = \frac{1}{2}\overrightarrow{\boldsymbol{dl}}\times\overrightarrow{\boldsymbol{dl}} = \hat{\boldsymbol{x}}\,dy\,dz + \hat{\boldsymbol{y}}\,dz\,dx + \hat{\boldsymbol{z}}\,dx\,dy$ $d\tau = \frac{1}{3}\overrightarrow{\boldsymbol{dl}}\cdot\overrightarrow{\boldsymbol{da}} = dx\,dy\,dz$

  $\nabla f = (\partial_x, \partial_y, \partial_z)f = (\hat{\boldsymbol{x}}\,\partial_x + \hat{\boldsymbol{y}}\partial_y + \hat{\boldsymbol{z}}\,\partial_z)f$ $\nabla\times\vec{\boldsymbol{A}} = (\partial_x, \partial_y, \partial_z)\times(A_x, A_y, A_z) = \begin{vmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z\end{vmatrix}$

  $\nabla\cdot\vec{\boldsymbol{B}} = (\partial_x, \partial_y, \partial_z)\cdot(B_x, B_y, B_z) = \partial_x B_x + \partial_y B_y + \partial_z B_z = B_{x,x} + B_{y,y} + B_{z,z}$

  $\nabla^2 f = (\partial_x, \partial_y, \partial_z)\cdot(\partial_x, \partial_y, \partial_z)f = \partial_x^2 f + \partial_y^2 f + \partial_z^2 f = f_{,xx} + f_{,yy} + f_{,zz}$

  Cylindrical coordinates (ρ, ϕ, z) $\vec{\boldsymbol{r}} = \hat{\boldsymbol{\rho}}\rho + \hat{\boldsymbol{z}}z$ $h_\rho = 1, h_\phi = \rho, h_z = 1$

  $\overrightarrow{\boldsymbol{dl}} = \hat{\boldsymbol{\rho}}\,d\rho + \hat{\boldsymbol{\phi}}\,\rho\,d\phi + \hat{\boldsymbol{z}}\,dz$ $\overrightarrow{\boldsymbol{da}} = \hat{\boldsymbol{\rho}}\,\rho\,d\phi\,dz + \hat{\boldsymbol{\phi}}\,dz\,d\rho + \hat{\boldsymbol{z}}\,\rho\,d\rho\,d\phi$ $d\tau = \rho\,d\rho\,d\phi\,dz$

  $\nabla f = \hat{\boldsymbol{e}}_i/h_i\;\partial_i f = (\hat{\boldsymbol{\rho}}\;\partial_\rho + \hat{\boldsymbol{\phi}}\;/\rho\;\;\partial_\phi + \hat{\boldsymbol{z}}\;\partial_z)f$ $\nabla\times\vec{\boldsymbol{A}} = \frac{1}{\rho}\begin{vmatrix}\hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \hat{\boldsymbol{z}} \\ \partial_\rho & \partial_\phi & \partial_z \\ A_\rho & \rho A_\phi & A_z\end{vmatrix}$

  $\nabla\cdot\vec{\boldsymbol{B}} = \rho^{-1}(\partial_\rho\rho B_\rho + \partial_\phi B_\phi) + \partial_z B_z$ $\nabla^2 f = (\rho^{-1}\,\partial_\rho\rho\,\partial_\rho + \rho^{-2}\,\partial_\phi^2 + \partial_z^2)f$

  Spherical coordinates (r, θ, ϕ) $\vec{\boldsymbol{r}} = \hat{\boldsymbol{r}}r$ $h_r = 1, h_\theta = r, h_\phi = rs_\theta$

  $\overrightarrow{\boldsymbol{dl}} = \hat{\boldsymbol{r}}\,dr + \hat{\boldsymbol{\theta}}\,r\,d\theta + \hat{\boldsymbol{\phi}}\,rs_\theta\,d\phi$ $\overrightarrow{\boldsymbol{da}} = \hat{\boldsymbol{r}}\,r^2 s_\theta\,d\theta\,d\phi + \hat{\boldsymbol{\theta}}\,rs_\theta d\phi\,dr + \hat{\boldsymbol{\phi}}\,r\,dr\,d\theta$ $d\tau = r^2 s_\theta\,dr\,d\theta\,d\phi$

  $\nabla f = (\hat{\boldsymbol{r}}\,\partial_r + \hat{\boldsymbol{\theta}}/r\;\partial_\theta + \hat{\boldsymbol{\phi}}/rs_\theta\;\partial_\phi)f$ $\nabla\times\vec{\boldsymbol{A}} = \varepsilon_{ijk}\,\hat{\boldsymbol{e}}_i/h_j h_k\;\;\partial_j h_k A_k = \frac{1}{r^2 s_\theta}\begin{vmatrix}\hat{\boldsymbol{r}} & r\hat{\boldsymbol{\theta}} & rs_\theta\hat{\boldsymbol{\phi}} \\ \partial_r & \partial_\theta & \partial_\phi \\ A_r & rA_\theta & rs_\theta A_\phi\end{vmatrix}$

  $\nabla\cdot\vec{\boldsymbol{B}} = (h_1h_2h_3)^{-1}\,\partial_k h_i h_j B_k = r^{-2}\partial_r r^2 B_r + r^{-1}s_\theta^{-1}(\partial_\theta s_\theta B_\theta + \partial_\phi B_\phi)$

  $\nabla^2 f = (h_1h_2h_3)^{-1}\partial_k\,h_i h_j/h_k\;\partial_k f = r^{-2}(\partial_r r^2\partial_r + s_\theta^{-1}\,\partial_\theta s_\theta\,\partial_\theta + s_\theta^{-2}\,\partial_\phi^2)f$

  $= r^{-1}\,\partial_r^2 r - r^{-2}L^2$ where $L^2 = -s_\theta^{-1}\,\partial_\theta s_\theta\,\partial_\theta - s_\theta^{-2}\,\partial_\phi^2$

- *Partial Differential Equations (eigenfunctions)*

  $\partial_x e^{ikx} = ik\,e^{ikx}$ $\partial_\phi e^{im\phi} = im\,e^{im\phi}$ $(-\partial_x(1-x^2)\,\partial_x + m^2/(1-x^2))P_\ell^m(x) = \ell(\ell+1)P_\ell^m(x)$

  $L^2 Y_{\ell m}(\theta,\phi) = \ell(\ell+1)Y_{\ell m}(\theta,\phi)$ $\partial_\phi Y_{\ell m}(\theta,\phi) = im\,Y_{\ell m}(\theta,\phi)$ $Y_{\ell m}(\theta,\phi) \propto P_\ell^m(c_\theta)e^{im\phi}$

  $(\rho^{-1}\partial_\rho\rho\partial_\rho - m^2/\rho^2)\,J_m(k_\rho\rho) = -k_\rho^2\,J_m(k_\rho\rho)$ $(r^{-2}\partial_r r^2\partial_r - \ell(\ell+1)/r^2)\,j_\ell(k_r r) = -k_r^2\,j_\ell(k_r r)$

  $(\nabla^2 + k^2)e^{i\vec{k}\cdot\vec{r}} = (\nabla^2 + k^2)J_m(k_\rho\rho)e^{im\phi}e^{ik_z z} = (\nabla^2 + k^2)j_\ell(k_r r)Y_{\ell m}(\theta,\phi) = 0$ $k^2 = \vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{k}} = k_x^2 + k_y^2 + k_z^2$

  $\nabla^2\,e^{kx+iky} = \nabla^2\,\rho^{\pm m}e^{im\phi} = \nabla^2\,r^{\ell,-\ell-1}\,Y_{\ell m}(\theta,\phi) = 0$ $= k_\rho^2 + k_z^2 = k_r^2$