University of Kentucky, Physics 306
Homework #1, Rev. B, due Wednesday, 2025-01-22

1. Perform the following exercises in **Mathematica**. The `Typewriter` font indicates command names.

a) `Plot Exp[-1/x]` and its `D`erivatives—it transitions very smoothly to 0 for all $x < 0$.

b) Find the `Series` of $(1+x)^a$ about $x = 0$ to deduce the binomial expansion.

c) Build a `Plot` of $x/(a^2 - x^2)$ piece by piece, starting from x^2, then $a^2 - x^2$, etc.

d) `Plot` the functions `Sin[x]`, `Sin[x-a]`, `Sin [x/b]`, `Sin[(x-a)/b]`, `Sin[x]^2`, `ArcSin[x]` using your own numbers for constants a and b.

e) Show graphically that i) $\cos^2(x) = (1+\cos(2x))/2$, ii) $\sin^2(x) = (1-\cos(2x))/2$, iii) $2\cos(x)\sin(x) = \sin(2x)$, iv) $\cos^2(x)+\sin^2(x) = 1$, v) $(\cos(x)-\sin(x))(\cos(x)+\sin(x)) = \cos(2x)$.

f) `Manipulate` a `Plot` of $f(t) =$ `x0 Exp[-alpha t] Sin[w t]` with sliders for the parameters x_0, α, ω to investigate their effect on damped oscillatory motion, and also $f(x,t) =$ `Cos[x - 2 Pi t]` versus x with sliding parameter t to see the wave travel.

g) `Solve x^2+1==0` for x and substitute `x->I` into the LHS $x^2 + 1$ using the `/.` operator.

h) Show Euler's theorem: `Exp[I x]==Cos[x] + I Sin[x]` by `Series` expansion of both sides. [*bonus*: `Solve` for $\cos(x)$ and $\sin(x)$ in terms of e^{ix} and e^{-ix}].

i) `Plot` the functions `Exp[x]/2, -Exp[x]/2, Exp[-x]/2, -Exp[-x]/2` along with the hyperbolic functions `Sinh[x], Cosh[x], -Sinh[x], -Cosh[x]` and compare your results with h).

j) Show that `Cosh[x]^2-Sinh[x]^2` $= 1$ and compare your results with e). [*bonus*: Parametrically plot (x,y)=`{Cos[t],Sin[t]}` and compare with `{Cosh[t],Sinh[t]}`. What is the relation to the above identities? Thus they are called circular (elliptical) and hyperbolic functions.]

k) Show graphically that `ArcSinh[y]` = `Log[y+Sqrt[y^2+1]]`. [*bonus*: What is `Tanh` in terms of `Exp`? Compare the *sigmoid* functions `Tanh[x]` and `2/Pi ArcTan[Pi/2 x]`.]

l) [*bonus*: Explore plots of the matrix equation `{x,y}.{{a,b},{b,c}}.{x,y}==0` for various values of (a, b, c) to discover conic sections. Calculate `Det[{{a,b},{b,c}}]` for each.]

m) [*bonus*: Normalize the normal distribution $p(x) =$ `Exp[-((x-mu)/sig)^2/2]` so that it `Integrate`s to 1 over the real line $-\infty < x < \infty$. Plot the normal distribution and the cumulative distribution $P(x) = \int_{-\infty}^{x} p(x)$ for $\mu = 2$ and $\sigma = 3$.]

n) [*bonus*: Show that the three functions $f(x,y) =$ `Cos[x]{Exp[-y],Sinh[y],Cosh[y]}` are all solutions of $\nabla^2 f = 0$. `ContourPlot` the functions.]

o) [*bonus*: `ContourPlot` the `Re`, `Im` parts of $f(z) = z^2 =$`(x + I y)^2`, `Cos[z]`, `Sin[z]`, `Exp[z]`.]

2. Taylor Series approximations allow focusing on the essential physics and analytically calculating problems with no closed form solution.

a) Calculate the Taylor series $f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{3!}f^{(3)}(a)(x-a)^3 + \ldots$ of $f(x) = e^x,\ \cos(x),\ \sin(x)$ to 5th order (x^5) about $a = 0$ and compare with Mathematica.

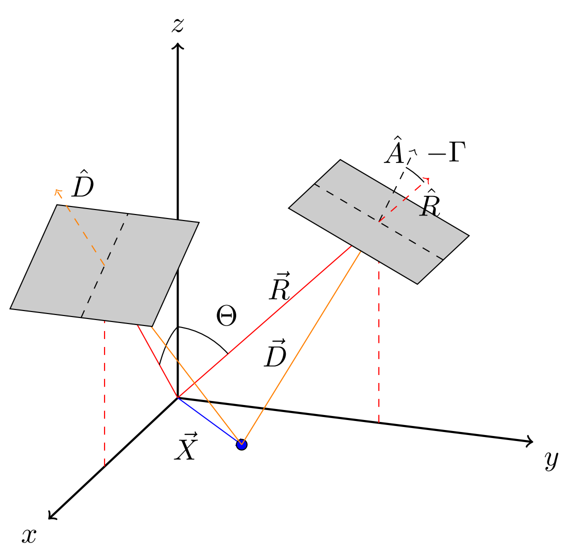
b) Perform long division of $\tan(x) = \sin(x)/\cos(x)$ from the series above to obtain the expansion for $\tan(x)$ to 5th order in x and compare with Mathematica.

c) Use the binomial series $(1+x)^\alpha = 1 + \frac{\alpha}{1}x + \frac{(\alpha)(\alpha-1)}{(1)(2)}x^2 + \ldots$ to obtain the **multipole expansion** of the electrostatic potential $V(\vec{\boldsymbol{r}}) = q/4\pi\epsilon \mathscr{r}$, where $\mathscr{r} = ((\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}') \cdot (\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}') = r^2 - 2rr'\cos\gamma + r'^2)^{1/2}$ to 3rd order in r'. Identify the first four *Legendre Polynomials* $P_\ell(\cos\gamma)$ in the Taylor expansion $V(\vec{\boldsymbol{r}}) = q/4\pi\epsilon r \sum_{\ell=0}^{\infty}(r'/r)^\ell P_\ell(\cos\gamma) = q/4\pi\epsilon(P_0(\cos\gamma) + P_1(\cos\gamma)(r'/r) + P_2(\cos\gamma)(r'/r)^2 + P_3(\cos\gamma)(r'/r)^3 + \ldots)$, which correspond to the monopole, dipole, quadrupole, and sextupole potentials [and moments].

d) In the small-area approximation, the **solid angle** subtended by a detector of area $|\vec{A}|$ with unit normal $\hat{A}$ located at spherical coordinates (R, Θ, Φ), tilted down by the angle Γ (from facing the origin) is

$$\Omega(x,y) = \frac{\vec{A} \cdot \hat{D}}{D^2} = \frac{A}{R^2} \frac{\cos\Gamma - \sin(\Theta + \Gamma)(x\cos\Phi + y\sin\Phi)}{(1 - 2\sin\Theta(x\cos\Phi + y\sin\Phi) + x^2 + y^2)^{3/2}}.$$

Expand $\Omega(x,y)$ to second order in x and y about the origin. [*bonus*: Show that for the total solid angle of 4 symmetric detectors at azimuthal angles $\Phi = 0, \pi/2, \pi, 3\pi/2$, all terms vanish except $\Omega(x,y) = \Omega_0 + \Omega_2(x^2 + y^2)$. Calculate the relation between Θ and Γ that forces $\Omega_2 = 0$]



A view of two detectors centered at $(R, \Theta, 0)$ and $(R, \Theta, \pi/2)$ in spherical coordinates.