University of Kentucky, Physics 306 Homework #3, Rev. A, due Wednesday, 2025-02-05

1. Orthogonal Projections: Parallel and perpendicular projections can be expressed as $P_{\parallel} = \hat{n}\hat{n} \cdot$ and $P_{\perp} = -\hat{n} \times \hat{n} \times$, in terms of the dot and cross products respectively, which themselves are projected products $\vec{a} \cdot \vec{v} = av_{\parallel}$ and $\vec{a} \times \vec{v} = \hat{n}av_{\perp}$. We will build them up geometrically piece-bypiece, and explore their matrix versions in this problem.

a) Starting with the vector $\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$, i) draw \vec{r} for some value of x and y, ii) illustrate the length $r_x = \hat{x} \cdot \vec{r}$, and the vector $\vec{r}_x = \hat{x}\hat{x} \cdot \vec{r}$ with the same magnitude, in the direction \hat{x} . iii) Calculate $\vec{r}_x = \hat{x}(\hat{x} \cdot \vec{r})$, where $\hat{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\hat{x} \cdot = \hat{x}^T = \begin{pmatrix} 1 & 0 \end{pmatrix}$. iv) Calculate the matrix $P_x = \hat{x}\hat{x}$ and then $\vec{r}_x = P_x\vec{r}$, which should be the same by associativity. v) Show that $P_x^2 = P_x$ algebraically and by multiplying matrices. vi) Calculate $P_y = \hat{y}\hat{y}$ and show that $P_x + P_y = I$, so that $\vec{r}_x + \vec{r}_y = \vec{r}$.

b) Starting with some vector $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, i) draw the vectors \vec{r} , $\hat{n}r_{\perp} = \hat{z} \times \vec{r}$ (projected and rotated by 90°), $-\vec{r}_{\perp} = \hat{z} \times (\hat{z} \times \vec{r})$ (another rotation), and \vec{r}_{\perp} , the projection in the *xy*-plane perpendicular to \hat{z} . ii) Calculate the components of $\vec{r}_{\perp} = -\hat{z} \times (\hat{z} \times \vec{r})$ iii) to obtain the matrix $P_{\perp} = -\hat{z} \times \hat{z} \times \hat{z}$ by its action on \vec{r} . iv) Show $P_{\perp}^2 = P_{\perp}$ by matrix multiplication and v) $P_{\parallel} + P_{\perp} = I$ by adding the matrices, and by using the rule $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$.

2. General Projections: relational versus parametric linear/planar geometry.

a) Show graphically that the following equations define the set of points $\{x\}$ on a line or plane,

	relational	parametric
line	$\{ec{x}\midec{a} imesec{x}=ec{d}\}$	$\{\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}_1 + \vec{\boldsymbol{a}}\alpha \qquad \alpha \in \mathbb{R}\}$
plane	$\{ ec{x} \mid ec{A} \cdot ec{x} = D \}$	$\{ec{m{x}}=ec{m{x}}_2+ec{m{b}}eta+ec{m{c}}\gamma\mideta,\gamma\in\mathbb{R}\}$

where $\vec{a}, \vec{d}, \vec{A}, D$ are constants which define the geometry, $\vec{x}_{1,2}$ are fixed points on the line and plane respectively, and α, β, γ are parameters that vary along the line/plane to uniquely parametrize points in the line/plane. [bonus: Show the 5th relation { $\vec{x} = \vec{x}_2 + \vec{A} \times \vec{\delta} | \vec{\delta} \in \mathbb{R}^3$ }.]

b) What constraint between \vec{a} and \vec{d} is implicit in the formula $\vec{a} \times \vec{x} = \vec{d}$? What is the relation between \vec{b} , \vec{c} , and \vec{A} ? Substitute \vec{x} of each parameterization into its relational equation to show the consistency between the two forms and to derive \vec{d} and D in terms of \vec{a} , \vec{x}_1 and \vec{A} , \vec{x}_2 .

c) Define $\tilde{a} \equiv \vec{A}/(\vec{a}\cdot\vec{A})$, which is parallel to \vec{A} , and *normalized* in the sense that $\vec{a}\cdot\tilde{a} = 1$. Using the BAC-CAB rule, show that $\vec{x} = \vec{a}(\tilde{a}\cdot\vec{x}) - \tilde{a}\times(\vec{a}\times\vec{x})$ for all \vec{x} . Illustrate this non-orthogonal projection of \vec{x} onto vectors \vec{x}_1 parallel to the line plus \vec{x}_2 parallel to the plane. [bonus: use this to calculate the point \vec{x}_0 at the intersection of the line and plane in terms of $\vec{a}, \vec{d}, \vec{A}, D$. Verify this by showing that \vec{x}_0 satisfies the relational equation for both the line and plane.]

d) Let $\tilde{a} \equiv \frac{\vec{b} \times \vec{c}}{\vec{a} \cdot \vec{b} \times \vec{c}}$, $\tilde{b} \equiv \frac{\vec{c} \times \vec{a}}{\vec{a} \cdot \vec{b} \times \vec{c}}$, and $\tilde{c} \equiv \frac{\vec{a} \times \vec{b}}{\vec{a} \cdot \vec{b} \times \vec{c}}$, where the arrows on $\vec{a}, \vec{b}, \vec{c}$ distinguish them from their *covectors* $\tilde{a}, \tilde{b}, \tilde{c}$. The definition of \tilde{a} is the same as in c) with $\vec{A} = \vec{b} \times \vec{c}$. Calculate the nine combinations of $(\vec{a}, \vec{b}, \vec{c})^T \cdot (\tilde{a}, \tilde{b}, \tilde{c}) = I$, i.e. $\vec{a} \cdot \tilde{a} = 1$, $\vec{a} \cdot \tilde{b} = 0$, ... to show they are *mutually*

orthonormal. In this sense, $(\tilde{a}, \tilde{b}, \tilde{c})$ is the <u>dual</u> basis of $(\vec{a}, \vec{b}, \vec{c})$. [bonus: Show using Cramer's rule that $[\tilde{a}|\tilde{b}|\tilde{c}] = [\vec{a}|\vec{b}|\vec{c}]^{-1}$, i.e. that it is a reciprocal basis.]

e) The contravariant components of \vec{x} are defined as the components (α, β, γ) that satisfy the equation $\vec{x} = \vec{a}\alpha + \vec{b}\beta + \vec{c}\gamma$; i.e., $(\vec{a}, \vec{b}, \vec{c})$ is the contravariant basis. Using $\vec{x} \cdot \tilde{a}$, etc., calculate the three contravariant components of \vec{x} in terms of dot products. Likewise, find the covariant components $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ of \vec{x} , defined as the components $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ which satisfy $\vec{x} = \tilde{a}\tilde{\alpha} + \tilde{b}\tilde{\beta} + \tilde{c}\tilde{\gamma}$; i.e. $(\tilde{a}, \tilde{b}, \tilde{c})$ is the covariant basis.