University of Kentucky, Physics 306
Homework #4, Rev. A, due Thursday, 2025-02-13

1. The **Pauli matrices**, defined as $\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, describe quantum mechanical spin $\frac{1}{2}$ particles in a space with two basis vectors representing spin up and down.

a) Show that $\sigma_j \sigma_k = \delta_{jk} I + i\varepsilon_{jk\ell}\sigma_\ell$. What is the *triple product* $\sigma_x\sigma_y\sigma_z$? Thus σ_j acts like the basis vectors $\hat{x}, \hat{y}, \hat{z}$, and encode the structure of the dot and cross products just as $1, i$ do in the complex plane. What are the inverses of σ_k?

[*bonus*: Show that product of *Dirac matrices* is $\gamma^\mu\gamma^\nu = g^{\mu\nu} I + i\sigma^{\mu\nu}$ and calculate the components $g^{\mu\nu}$ and the matrices $\sigma^{\mu\nu}$. They encode the structure of spacetime in special relativity.]

b) Show that $(I, \sigma_x, \sigma_y, \sigma_z)$ is a basis of the generalized vector space of complex 2×2 matrices: they are independent and any matrix can be written as the linear combination $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = IA_0 + \sigma_x A_x + \sigma_y A_y + \sigma_z A_z$ by solving for A_0, A_x, A_y, A_z.

What are the original basis matrices $P_{1,2}$ and $\sigma_\pm$, such that $A = P_1 a + \sigma_+ b + \sigma_- c + P_2 d$?

c) Verify that the transformation matrices $U_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $U_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ are *unitary*, $U_x^\dagger U_x = U_y^\dagger U_y = I$, so that $U_x^{-1} = U_x^\dagger$ and $U_y^{-1} = U_y^\dagger$.

d) Verify the *eigenequations* $\sigma_x U_x = U_x \sigma_z$ and $\sigma_y U_y = U_y \sigma_z$. Calculate the *similarity transform* $U_x \sigma_k U_x^{-1}$ of each of the Pauli matrices $\sigma_k = \sigma_x, \sigma_y, \sigma_z$. For $i, j{=}x, y, z$, find the matrices U_{ij} that transform σ_i into $\sigma_j = U_{ij}\sigma_i U_{ij}^{-1}$. by rearranging the eigenequations above. In these equations, $_{i,j}$ are not component indices, but label (distinguish) different U matrices.

e) The *trace* of a matrix is the sum of its diagonal elements: $\mathrm{tr}A = A_{ii}$. Calculate $\mathrm{tr}(0)$ (the zero matrix) and $\mathrm{tr}(\sigma_i)$. [*bonus*: Show that $\mathrm{tr}(ABC) = \mathrm{tr}(BCA) = \mathrm{tr}(CAB)$ but not necessarily $\mathrm{tr}(ABC) = \mathrm{tr}(CBA)$.] Use this to show that if $A' = UAU^\dagger$, then $\mathrm{tr}(A') = \mathrm{tr}(A)$, i.e. the trace, or 'perimeter', of A is invariant.

f) The *determinant* of a matrix is the fully antisymmetric product of one element in each row or column: $\det A = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k} = \varepsilon_{ijk} A_{i1} A_{j2} A_{k3}$, which has the properties $\det(A^\dagger) = \det(A)^*$ and $\det(AB) = \det(A)\det(B)$. Calculate $\det(I)$ and $\det(\sigma_i)$. Show that if $U^\dagger U = I$ then $|\det(U)| = 1$, and if $A' = UAU^\dagger$, then $\det(A') = \det(A)$, i.e. the determinant, or 'volume', of A is invariant.

2. Rotations of the vector $\vec{\boldsymbol{v}} = \begin{pmatrix} x \\ y \end{pmatrix}$ are generated by the matrix $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which represents a 90° CCW rotation—just as complex rotations $e^{i\phi}$ are generated by i, where $i^2 = -1$.

a) Show that $M\boldsymbol{v}$ rotates $\boldsymbol{v}$ by 90° CCW, and that $M^2 = -I$.

b) Show that $R = e^{M\phi} = I\cos\phi + M\sin\phi$ and calculate the components of R. Show that $M^T = -M$ (the generator is asymmetric) and thus $R^T R = I$ (verify it), just like $(e^{i\phi})^*(e^{i\phi}) = 1$. Show that $dR = MRd\phi$, just like $de^{i\phi} = ie^{i\phi}d\phi$ in H02#1d.

c) The cross product $\times$ generates rotation in 3d. Calculate the matrices $\boldsymbol{M} = (M_x, M_y, M_z)$, where $\hat{\boldsymbol{x}} \times \boldsymbol{v} = M_x \boldsymbol{v}$, etc. Show that the components of M_ℓ are $(M_\ell)_{jk} = \varepsilon_{kj\ell}$ and that $M_\ell^2 = -P_{\perp\ell}$, where $P_{\perp\ell}$ projects perpendicular to $\hat{\boldsymbol{e}}_\ell$. Thus the general matrix for a CCW rotation of angle v about the $\hat{\boldsymbol{v}}$-axis is $R_{\boldsymbol{v}} = e^{\boldsymbol{M}\cdot\boldsymbol{v}} = P_\perp \cos v + \boldsymbol{M}\cdot\hat{\boldsymbol{v}} \sin v + P_\parallel = I \cos v + \boldsymbol{M}\cdot\hat{\boldsymbol{v}} \sin v + \hat{\boldsymbol{v}}\hat{\boldsymbol{v}}^T(1-\cos v)$, *Rodrigues' formula*. The third term preserves the projection along the axis of rotation $\hat{\boldsymbol{v}}$. Verify this formula for the familiar case $\vec{\boldsymbol{v}} = \hat{\boldsymbol{z}}\phi$.

d) [*bonus*: The Pauli matrices also generate rotation. Calculate $e^{-i\sigma_x\phi}$, $e^{-i\sigma_y\phi}$, $e^{-i\sigma_z\phi}$. Which transformation corresponds to R and the complex rotation $e^{i\phi}$?]

[*bonus*: **3. Minkowski space**. We will derive the Lorentz transformations using the Lorentz metric, which encodes Einstein's principle of special relativity that the speed of light c is constant in any reference frame.

a) Show that the principle of relativity can be written as $\boldsymbol{x}^2 = \boldsymbol{x}\cdot\boldsymbol{x} = \boldsymbol{x}^T g \boldsymbol{x} = 0$ for the *space-time vector* $\boldsymbol{x} = (ct\ x)^T$ with components representing the distance $x = ct$ traveled by a photon in time t, where the matrix $g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the *Minkowski metric*. In three-dimensional space, $\boldsymbol{x} = (ct\ x\ y\ z)^T$ is a 4-vector, and the metric becomes a 4×4 matrix $g = \mathrm{diag}(-1,1,1,1)$.

b) Momentum and energy can also be combined into the space-time vector $\boldsymbol{p} = (E/c\ p)^T$. Show that invariance of the dot product $\boldsymbol{p}^T g \boldsymbol{p} = -(mc)^2$ leads to the formula $E^2 = (pc)^2 + (mc^2)^2$, which reduces to Einstein's equation $E = mc^2$ when $p = 0$.

c) Normal rotations R keep the length of vector constant by preserving the Euclidean metric I: $R^T I R = I$. Likewise, Lorentz transformations Λ preserve the Minkowski metric: $\Lambda^T g \Lambda = g$. For a small 'rotation' $\Lambda = I + G d\alpha$ generated by G, show that $G^T g + gG = 0$. Show that $G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ satisfies this relation to first order in $d\alpha$. Note the difference between G and M!

d) Show that $G^2 = I$ and therefore $\Lambda = e^{G\alpha} = I\cosh\alpha + G\sinh\alpha = \gamma\begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}$, where $\beta = \tanh\alpha = v/c$ and $\gamma = \cosh\alpha = (1-\beta^2)^{-1/2}$. Thus the Lorentz transformations $\boldsymbol{x}' = \Lambda\boldsymbol{x}$ are $t' = \gamma(t + vx/c^2)$ and $x' = \gamma(x + vt)$, which encode all of the features of special relativity. Verify that $\Lambda^T g \Lambda = g$ and plot the rotated basis vectors. Calculate the relativistic addition rule for velocities β_1 and β_2 from $\Lambda = \Lambda_2\Lambda_1$.

e) The Dirac matrices γ^μ generate Lorentz rotations. Calculate $e^{-i\gamma_\mu v^\mu}$.]