University of Kentucky, Physics 306 Homework #5, Rev. A, due Monday, 2025-02-18

1. Stretches. In analogy with the *polar decomposition* $w = x + iy = \rho e^{i\phi}$ of complex numbers, any matrix A can be decomposed A = RS into a stretch S and a rotation R, which are the building blocks of all linear operators. This problem set explores the structure of stretches.

a) For any eigenvalue problem $M\vec{v}_i = \vec{v}_i\lambda_i$, augment these *n* equations to obtain MV = VW, where $W = \text{diag}(\lambda_1, \lambda_2, \ldots)$, and thus show $M = VWV^{-1}$ and $W = V^{-1}MV$. This is the *similarity* transform to the eigenbasis of M, in which the operator's matrix is diagonal, and its inverse.

b) Diagonalize σ_x and σ_y by calculating their eigenbasis $U_{x,y} = (\vec{v}_1 | \vec{v}_2)$ of H04#1c to justify the similarity transforms $\sigma_x U_x = U_x \sigma_z$ of H04#1d. Thus, $\sigma_{x,y}$ are matrices of the same operator σ_z in different bases. Calculate the eigenvalues and eigenvectors of σ_z , which is already diagonal.

c) Calculate all eigenvalues and eigenvectors of the ladder operators $\sigma_{\pm} = (\sigma_x \pm i\sigma_y)/2$ of H04#1b to show that they are not diagonalizable. Such matrices with less eigevectors than eigenvalues are called *defective* (see H08#2). Show that they are also *nilpotent*: $\sigma_{\pm}^n = 0$ for some value n.

d) Calculate the eigenvalues and eigenvectors of the projections $P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ of H04#1b. [bonus: show that projections, $P^2 = P$, can only have eigenvalues of 0 or 1.]

e) A symmetric matrix S is guaranteed to have a complete set of eigenvectors $V = (\vec{v}_1, \vec{v}_2, ...)$ and corresponding eigenvalues $\lambda_1, \lambda_2, ...$, such that $S\vec{v}_i = \vec{v}_i\lambda_i$ with the following special properties: show that i) the eigenvalues of S must be real: $\lambda^* = \lambda$, and that ii) two eigenvectors \vec{v}_i, \vec{v}_j of S with distinct eigenvalues $\lambda_i \neq \lambda_j$ must be orthogonal: $\vec{v}_i \cdot \vec{v}_j = 0$. Thus the matrix of eigenvectors is unitary: $V^{\dagger}V = I$, so that $S = VWV^{\dagger}$ and $W = V^{\dagger}SV$. Interpret these similarity transforms geometrically.

f) In addition to symmetric matrices, complex normal matrics, $N^{\dagger}N = NN^{\dagger}$ (see H08#1), also have an othogonal eigenbasis but only have real eigenvalues if they are Hermitian, $H^{\dagger} = H$. Calculate the eigenvalues and eigenvectors of $M_z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. How were these used previously? Show that $M_z = VWV^{\dagger} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ and $e^{M_z\phi} = Ve^{W\phi}V^{\dagger} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$. Multiply this out to verify H04#2b. This is an example of the normal matrix analogy, which relates matrices have zeros on the diagonal and imaginary eigenvalues. The exponential of a Hermitian matrix is positive definite and has real positive eigenvalues, while the exponential of an anti-Hermitian matrix is unitary with unit modulus eigenvalues and therefore determinant.

2. [bonus: We saw in H04#1 that the trace and determinant are matrix **invariants** under similarity transformation. In general, an $n \times n$ matrix has n independent invariants, including the trace and determinant.

a) Show that the characteristic equation $|A - \lambda I| = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$, substituting $\lambda \to A$ in the second equality, is invariant under similarity transforms. Thus A has n independent invariants: either $a_0 \ldots a_{n-1}$ of this equation, its n eigenvalue roots, or the traces of A^1, A^2, \ldots, A^n . The first and last coefficients a_{n-1} and a_0 are tr(A) and det(A), respectively, while the others are k-dimensional 'perimeters', for example, surface area, of the transformation. b) Prove the Cayley-Hamilton Theorem, that any matrix A satisfies its own characteristic equation, for the case of diagonal matrices. The full theorem follows from the invariance of the characteristic equation. Show that $A^{-1} = -(A^{n-1} + a_{n-1}A^{n-2} + \ldots + a_1I)/a_0$.]