University of Kentucky, Physics 306
Homework #7, Rev. A, due Monday, 2025-03-10

1. The **Legendre Polynomials** $P_\ell(x)$ appear in the solution of Laplace's equation in spherical coordinates. They can be defined as *the* series of orthogonal polynomials of degree $\ell = 0, 1, 2, \ldots$ with respect to the weight $w(x) = 1$ over the range $-1 \le x \le 1$, *standardized* to $P_\ell(1) = 1$.

a) Show that $\langle x^m | x^n \rangle = \frac{2}{m+n+1}$ [$m+n$ is even] or 0 [$m+n$ is odd]. Apply the Gram-Schmidt procedure to the basis of monomials $1, x, x^2, \ldots$ to generate the series of $P_\ell(x)$ up to order $\ell = 4$. Remmeber to standardize, not normalize, each basis function, so that $P_\ell(1) = 1$. Calculate the *normalization* $h_\ell^2 = \langle P_\ell | P_\ell \rangle$. Expand $f(x) = x^4$ in the Legendre polynomial basis.

b) Show that these basis functions $P_\ell(x)$ are eigenfunctions of the linear Legendre differential operator $L = -\frac{d}{dx}(1-x^2)\frac{d}{dx}$ by direct calculation of their eigenvalues $L[P_\ell(x)] = \lambda P_\ell(x)$.

2. The **Sturm-Liouville** 2nd order linear differential operator, which generalizes 1b) to

$$L[u(x)] \equiv \frac{1}{w(x)}\left[\frac{d}{dx}p(x)\frac{d}{dx} - q(x)\right]u(x), \tag{1}$$

is self-adjoint, $L^\dagger = L$, [it's a **stretch**!] with respect to the inner product

$$\langle u_1 | u_2 \rangle \equiv \int_a^b w(x)dx\, u_1^*(x)\, u_2(x) \tag{2}$$

if we impose the boundary conditions $u(a)w(a) = u(b)w(b) = 0$. Thus L has real eigenvalues λ_i and a complete set of orthogonal eigenfunctions. Any function $f(x)$ has the expansion $|f\rangle = \sum_i |u_i\rangle f_i$ or $f(x) = \sum_i u_i(x) f_i$, with components $f_i = \langle u_i | f \rangle / \langle u_i | u_i \rangle = \int_a^b w(x)dx\, u_i^*(x) f(x)/h_i^2$.

a) Show that L is *self-adjoint* or Hermitian. *Hint:* use the definition $\langle f | H^\dagger g \rangle \equiv \langle Hf | g \rangle$ to show that the operator $D = \frac{1}{w}\frac{d}{dx}$ is anti-Hermitian and apply it to the composition of operators in L. [*bonus*: Show that any linear 2^{nd}-order differential operator over any weight can be put in this form.]

b) Given eigenfunctions $L|u_i\rangle = \lambda_i |u_i\rangle$, i) show that $\lambda_i \in \mathbb{R}$ and that $\langle u_i | u_j \rangle = 0$ for all $\lambda_i \neq \lambda_j$. ii) Show the *closure* relation $I = \sum_i |u_i\rangle\langle u_i|/h_i^2$ by finding the components f_i of the linear combination $|f\rangle = \sum_i |u_i\rangle f_i$ in the basis $|u_i\rangle$ and substituting f_i back into this expansion to obtain an equation of the form $|f\rangle = I|f\rangle$. iii) Operate L on the same expansion to show that its *spectral decomposition* is $L = \sum_i \lambda_i |u_i\rangle\langle u_i|/h_i^2$. From ii), what are the eigenvalues of the identity I?

c) The Legendre polynomials $P_\ell(\cos\theta)$, used with the polar coordinate θ in spherically symmetric PDEs, are eigenfunctions of the operator $L = \frac{d^2}{d\theta^2} + \cot\theta\frac{d}{d\theta}$. Show that this is a Sturm-Liouville system on the domain $0 < \theta < \pi$, with $w(\theta) = \sin\theta$, $p(\theta) = \sin\theta$, and $q(\theta) = 0$. Change variables to $x = \cos\theta$ and calculate the new functions $w(x)$, $p(x)$, $q(x)$ and domain $a < x < b$ and compare with problem #1b). Note the sign change and reversal of integration limits!

d) [*bonus*: Compile a chart of $w(x), a, b, p(x), q(x), \lambda_i, h_i^2$ for each of the following *orthogonal functions*: i) cylindrical harmonics $e^{im\phi}$; ii) associated Legendre functions $P_\ell^{|m|}(\cos\theta)$; iii) Fourier series $\sin(k_n x)$; iv) Bessel functions $J_m(k_{nm}\rho)$; v) spherical Bessel functions $j_\ell(k_{n\ell}r)$; vi) Hermite polynomials $H_n(x)$; vii) associated Laguerre polynomials $L_n^{(\alpha)}(x)$; viii) Airy functions $\mathrm{Ai}(\mathrm{x}+\mathrm{x}_0)$. (see the NIST Digital Library of Mathematical Functions)]