University of Kentucky, Physics 306
Homework #9, Rev. A, due Wednesday, 2025-03-31

1. Vectors in **curvilinear coordinates** (q^1, q^2, q^3) have a natural *coordinate basis* $\boldsymbol{b}_i \equiv \partial\boldsymbol{r}/\partial q^i$ and *reciprocal basis* $\boldsymbol{b}^i \equiv \boldsymbol{\nabla} q^i = \partial q^i/\partial\boldsymbol{r}$. Each basis vector is a *vector field* (a function of position). The most common coordinate systems are Cartesian $q^i = (x, y, z)$, cylindrical $q^i = (\rho, \phi, z)$, and spherical $q^i = (r, \theta, \phi)$, defined by the transformations $x + iy = \rho e^{i\phi}$ and $z + i\rho = re^{i\theta}$, respectively. These are all orthogonal, right-handed systems, for which both bases are aligned with the *orthonormal basis* $\hat{\mathbf{e}}_i = \boldsymbol{b}_i/h_i = \boldsymbol{b}^i h_i$, where $h_i = |\boldsymbol{b}_i| = 1/|\boldsymbol{b}^i|$ is called the *scale factor*.

a) Determine the coordinate transformation $q^i(q^{i'})$ from each coordinate system to each of the others. *Hint: invert and combine the two transformations above.*

b) For each coordinate system, illustrate the three coordinate isosurfaces $q^i(\boldsymbol{r}) = q^i_0$ (constant) passing through an arbitrary point $\boldsymbol{r}_0$, labeling lengths and angles in your diagram. For each coordinate q^i, identify the curve $\boldsymbol{s}(q^i; q^j_0, q^k_0)$ at the intersection of two surfaces of constant $q^j = q^j_0$ and $q^k = q^k_0$.

c) For each coordinate system, calculate $\boldsymbol{b}_i = \partial\boldsymbol{r}/\partial q^i$ using $\boldsymbol{dr} = \hat{\boldsymbol{x}}dx + \hat{\boldsymbol{y}}dy + \hat{\boldsymbol{z}}dz$. Calculate the *metric* $g_{ij} = \boldsymbol{b}_i \cdot \boldsymbol{b}_j = \mathrm{diag}(h_1^2, h_2^2, h_3^2)$ and normalize $\boldsymbol{b}_i = \hat{\mathbf{e}}_i h_i$ to find the *unit vectors*. The scale factors h_θ and h_ϕ for angular coordinates are just the radii of curvature, according to the arc length formulae $ds_\theta = rd\theta$ and $ds_\phi = \rho d\phi$.

d) Construct the transformation matrices between unit bases, by considering rotations $R_{\hat{z}}(\phi)$ (rotation by an angle ϕ about the z-axis) and $R_{\hat{\phi}}(\theta)$ (about the rotated y-axis). Compare with part c).

e) For each coordinate system, calculate the *line element* $\boldsymbol{dl} = \hat{\mathbf{e}}_i h_i dq^i$, the *area element* $\boldsymbol{da} = \frac{1}{2}\boldsymbol{dl} \times \boldsymbol{dl} = \hat{\mathbf{e}}_k\, h_i h_j\, dq^i dq^j$, and the *volume element* $d\tau = \frac{1}{3}\boldsymbol{dl} \cdot \boldsymbol{da} = h_1 h_2 h_3\, dq^1 dq^2 dq^3$.

f) Use the coordinate transformations $(\rho, \phi, z) = f^{-1}(x, y, z)$ from a) to calculate the covariant basis $\boldsymbol{b}^i = \nabla q^i = \hat{\mathbf{e}}_i/h_i$ and verify that $\boldsymbol{b}_i \cdot \boldsymbol{b}^j = \delta^j_i$. Calculate $g^{ij} = \boldsymbol{b}^i \cdot \boldsymbol{b}^j = \mathrm{diag}(h_1^{-2}, h_2^{-2}, h_3^{-2})$. [*bonus*: Do the same for spherical coordinates.]

[*bonus:* **2. Conformal maps**: in contrast to vector spaces, which have a linear (parallel) structure, curvilinear coordinate systems (parametrizations of points in space) can have curves and surfaces of any nondegenerate shape (every point has unique coordinates, except possibly a few singularities). A coordinate transformation is a multidimensional function $f : (x, y, z) \to (u, v, w)$ or its inverse $f^{-1} : (u, v, w) \to (x, y, z)$ between two coordinates of the same point. In 2d these can be represented by complex functions $w = u + iv = f(x + iy) = f(z)$.

An orthogonal coordinate system is one in which the coordinate lines and surfaces intersect at right angles. In 2d, this is automatically satisfied by *analytic* functions, which formally depend only on z and not z^*, for example, any combination of algebraic or trigonometric functions of z. In fact, both $u(x, y)$ and $v(x, y)$ satisfy the Laplace equation, so that the contours of these function represent physical potentials and field lines, respectively in 2d. These functions can be composed to create most common coordinate systems in 2d.

For each of the functions $w = f(z)$ mapping $z = x + iy \mapsto w = u + iv$, plot: i) the contours of $u(x, y)$ and $v(x, y)$ in the z-plane; and ii) the two families of curves $f(x + iy)$, parameterized by constant x or y respectively, in the w-plane (the inverse transformation).

a) $f(z) = z + c$ for a complex constant $c = a + ib$ (use $c = 2 + i$).

b) $f(z) = cz$ for the same constant c.

c) $f(z) = z^2$. Show the two branches in the z-plane and the corresponding branch cut in w.

d) $f(z) = e^z$, the transformation to polar coordinates $z = \rho e^{i\phi}$.

e) $f(z) = \cosh(z)$, the transformation to elliptical coordinates.

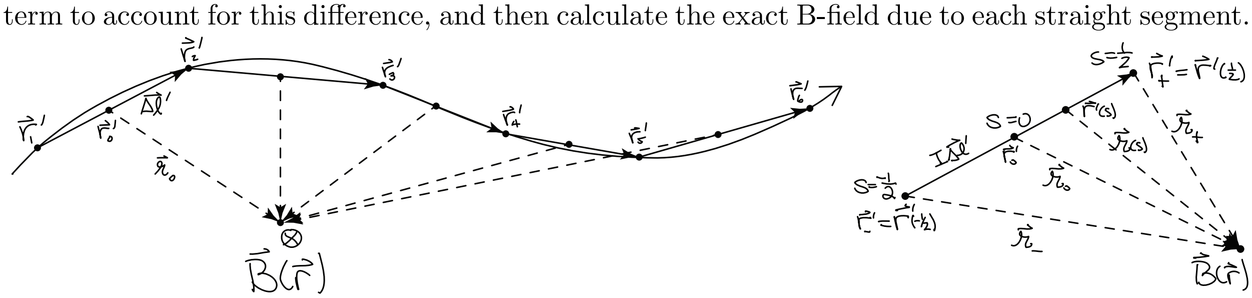
f) $f(z) = e^z + z$. A contour of this function was used by Rogowski to create a smooth edge of and electrode without any "hot spots" of high electric field which would arc. The other family of contours represents the lines of electric flux ending at charges on the electrode.

g) Conformal maps are ideal candidates for orthogonal coordinate systems, and can be used to succinctly represent coordinate transformations. For example, the transformation between Cartesian and cylindrical coordinates is simply the rotation $x + iy = \rho e^{i\phi}$ with z the same, and the transformation from cylindrical to spherical is the second rotation $z + i\rho = re^{i\theta}$ with ϕ the same. Write each of the orthogonal coordinate systems of Spiegel, "Vector Analysis", Ch. 7, in terms of conformal functions $(u, v, w) \mapsto (x, y, z)$: 1) cylindrical, 2) spherical, 3) parabolic, 4) paraboloidal, 5) elliptic, 6) prolate spheroidal, 7) oblate spheroidal, 8) ellipsoidal, 9) bipolar.]

3. The magnetic analog of *Coulomb's law* (with a scalar charge element $dq = \lambda dl = \sigma da = \rho d\tau$) is the **Biot-Savart law** (with a vector current element $\boldsymbol{v}dq = I\boldsymbol{dl} = \boldsymbol{K}da = \boldsymbol{J}d\tau$):

$$\boldsymbol{B} \;=\; \frac{\mu_0}{4\pi}\oint' \frac{\boldsymbol{v}dq' \times \boldsymbol{\mathscr{r}}}{\mathscr{r}^3} \;\approx\; \sum_i \left(\Delta\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{I\Delta\boldsymbol{\ell} \times \boldsymbol{\mathscr{r}}_0}{\mathscr{r}_0^3}\right)_i, \tag{1}$$

where $\Delta\boldsymbol{\ell}$ is the displacement vector from the beginning to the end of each current segment, and $\boldsymbol{\mathscr{r}}_0$ is the displacement vector from the middle of each current segment $\boldsymbol{r}'_0$ to the field point $\boldsymbol{r}$. The approximation is that all of the current is concentrated at $\boldsymbol{r}'_0$ instead of spread out along the length of the segment from $\boldsymbol{r}'_0 - \Delta\boldsymbol{\ell}/2$ to $\boldsymbol{r}'_0 + \Delta\boldsymbol{\ell}/2$. In this problem we first calculate a correction term to account for this difference, and then calculate the exact B-field due to each straight segment.



a) To analytically integrate the Biot-Savart law along a single straight segment of the path, parametrize the segment $\boldsymbol{r}'(s)$ with the parameter s, ranging from $s = -\frac{1}{2}$ at the beginning to $s = +\frac{1}{2}$ at the end of the segment. The parameterization involves the constant vectors $\boldsymbol{r}'_0$ (the center of the segment) and $\Delta\boldsymbol{\ell}$ (displacement along the segment). Calculate the line element $\boldsymbol{dl} = \frac{d\boldsymbol{r}'}{ds}ds$. Calculate $\boldsymbol{\mathscr{r}}$ as a function of $\boldsymbol{\mathscr{r}}_0$, $\Delta\boldsymbol{\ell}$, and s. Substitute these into the Biot-Savart formula and factor out the constant approximation of Eq. 1] to obtain

$$\Delta\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu 0}{4\pi}\frac{I\Delta\boldsymbol{\ell}\times\boldsymbol{\mathscr{r}}_0}{\mathscr{r}_0^3}T(\alpha, \beta), \tag{2}$$

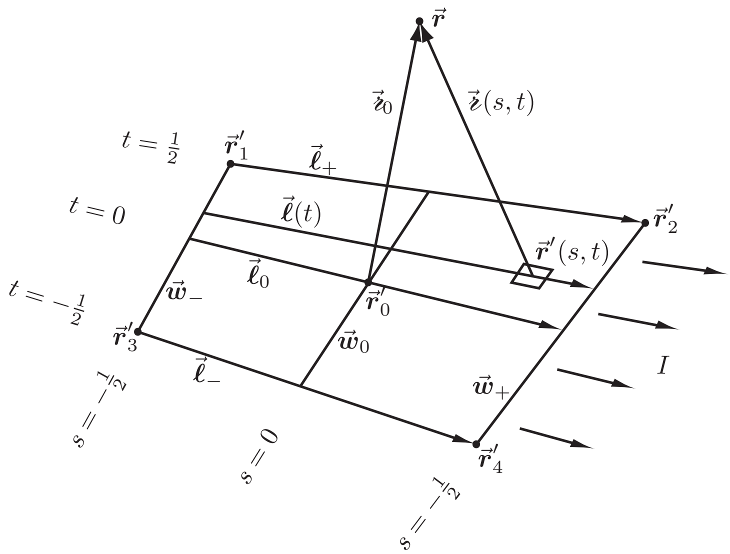
where the integral $T(\alpha, \beta)$ along s depends on $\alpha = \boldsymbol{\mathscr{r}}_0 \cdot \Delta\boldsymbol{\ell}/\mathscr{r}_0^2$ and $\beta = \Delta\ell^2/\mathscr{r}_0^2$.

b) [bonus: Approximate the integrand of $T(\alpha,\beta)$ to order s^2 and integrate to obtain the correction term $T(\alpha,\beta) \approx 1 + \frac{1}{8}(5\alpha^2 - \beta)$ for the case where all the current is at the center of the segment.]

c) Calculate the exact integral $T(\alpha,\beta)$ and show that $\Delta\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\frac{\Delta\boldsymbol{\ell}\times\boldsymbol{\mathscr{r}}_0}{(\Delta\boldsymbol{\ell}\times\boldsymbol{\mathscr{r}}_0)^2}\,\Delta\boldsymbol{\ell}\cdot(\hat{\boldsymbol{\mathscr{r}}}_- - \hat{\boldsymbol{\mathscr{r}}}_+)$, where $\boldsymbol{\mathscr{r}}_\pm = \boldsymbol{r} - \boldsymbol{r}'(\pm\frac{1}{2})$ is the displacement vector from each end of the segment to the field point and $\hat{\boldsymbol{\mathscr{r}}}_\pm = \boldsymbol{\mathscr{r}}_\pm/\mathscr{r}_\pm$.

d) [bonus: show that this is equivalent to $\Delta\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\frac{(\boldsymbol{\mathscr{r}}_-\times\boldsymbol{\mathscr{r}}_+)(\mathscr{r}_-+\mathscr{r}_+)}{\mathscr{r}_-\mathscr{r}_+\,(\mathscr{r}_-\mathscr{r}_+ + \boldsymbol{\mathscr{r}}_-\cdot\boldsymbol{\mathscr{r}}_+)}$.]

[*bonus:* **4. Current sheet**—surface currents can be approximated numerically by a tiling of quadrilaterals like the one shown below, with current I flowing parallel to the top and bottom edges, from left to right. Let the vector $\boldsymbol{\ell} = \boldsymbol{\ell}_+ = \boldsymbol{\ell}_-$ run along either the top or bottom edge, parallel to the current; $\boldsymbol{w} = \boldsymbol{w}_- = \boldsymbol{w}_+$ from bottom to top along the left or right edge; and $\boldsymbol{r}_0$ be the point at the center of the parallelogram as shown in the diagram.



a) Parametrize the surface of the parallelogram as $\boldsymbol{r}'(s,t)$, with the top and bottom edges are at $t = +\frac{1}{2}$ and $-\frac{1}{2}$, and the left and right edges are at $s = +\frac{1}{2}$ and $-\frac{1}{2}$ respectively.

b) Write down the Biot-Savart integral for the magnetic field in terms of integration parameters s, t and constants $\boldsymbol{\mathscr{r}}_0 \equiv \boldsymbol{r} - \boldsymbol{r}'_0$, $\boldsymbol{\ell}$, and $\boldsymbol{w}$.

c) Expand in powers of s and t, to calculate the integral up to second order.]

d) It is not possible to tile arbitrary surfaces with parallelograms—we need all four points on the quadrilateral to be arbitrary. To generalize this solution, let $\boldsymbol{\ell}_0$, $\boldsymbol{w}_0$ be the corresponding vectors through the center of the quadrilateral. We need one more vector $\boldsymbol{u}_0 = \boldsymbol{\ell}_+ - \boldsymbol{\ell}_- = \boldsymbol{w}_+ - \boldsymbol{w}_-$, where $\boldsymbol{\ell}_\pm$ run across the top and bottom, and $\boldsymbol{w}_\pm$ run along the right and left sides of the diagram. Generalize steps (a)-(c) to calculate $\boldsymbol{B}(\boldsymbol{r})$.]