University of Kentucky, Physics 306
Homework #12, Rev. A, due Wednesday, 2024-04-28

1. Solve Laplace's equation $\nabla^2 V = (\partial_x^2 + \partial_y^2)V = 0$ for the **tent potential** $V(x, y)$ defined on the region $-a<x<a$ and $-b<y<b$ with boundary conditions $V(x, \pm b) = 0$ and $V(\pm a, y) = V_0(1-|y/b|)$ by following these steps:

a) Substitute the eigenfunctions $X(x)$, $Y(x)$ and eigenvalues $-k_x^2$, $-k_y^2$ of the two differential operators ∂_x^2, ∂_y^2 of Laplace's equation to get the *disperson relation* between the two eigenvalues.

b) Apply boundary conditions at $y = \pm b$ to quantize k_n, $Y_n(y)$ and therefore $X_n(x)$, and form a linear combination of all eigenfunctions to obtain the general solution $V(x, y) = \sum_n c_n X_n(x) Y_n(y)$.

c) Apply boundary conditions at $x = \pm a$ to solve the coefficients c_n of the general solution.

d) Sketch the solution and its first three Fourier components in Mathematica.

2. Drumhead waves are described by the PDE $(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla_\perp^2)\eta(\rho, \phi, t) = 0$, where the wave velocity $v = \sqrt{\gamma/\sigma}$ depends on the surface tension γ and the mass density σ of the drumhead.

a) Use $\partial_t e^{-i\omega t} = -i\omega\, e^{-i\omega t}$ to obtain the Helmholtz equation $(\nabla_\perp^2 + k^2)\eta = 0$ by replacing ∂_t with its *eigenvalue*. Determine the *dispersion relation* between spatial k and temporal ω frequencies.

b) Expand $\nabla_\perp^2 \eta$ in cylindrical coordinates and show the radial part has the equivalent forms

$$\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} \quad = \quad \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} \quad = \quad \frac{1}{\sqrt{\rho}}\frac{\partial^2}{\partial \rho^2}\sqrt{\rho} + \frac{1}{4\rho^2}$$

c) Use the eigenvalue equation $\partial_\phi \Phi_m(\phi) = im\,\Phi_m(\phi)$ to factor out the ϕ dependence in the Laplacian and obtain the *Bessel equation*. Plot the first three *Bessel functions* $J_0(x)$, $J_1(x)$, and $J_2(x)$, where $x = k\rho$. Find the lowest-order Taylor approximation of each function as $x \to 0$ and the asymptotic approximation as $x \to \infty$. The energy, which is proportional to the amplitude η squared, spreads out as the circular wavefront expands.

d) Use the boundary conditions $\eta(\rho, 0) = \eta(\rho, 2\pi)$ and $\eta_{,\phi}(\rho, 0) = \eta_{,\phi}(\rho, 2\pi)$ to show that m must be an integer. Use the linearity of ∂_ϕ on $\Phi_m(\phi) \pm \Phi_{-m}(\phi)$ to show that $\cos(m\phi)$ and $\sin(m\phi)$ are also eigenfunctions of ∂_ϕ^2 (but not ∂_ϕ—why?) and determine the eigenvalues. Apply the boundary condition $\eta(a, \phi) = 0$ to find the possible values of k, in terms of x_{nm}, the n^{th} zero of the Bessel function $J_m(x)$. For each combination of m, n plot the *node lines* where $\eta_{mn}(\rho, \phi) = 0$ and find the vibrational frequency ω_{mn} of this mode.

e) [*bonus*: how could this solution be modified to solve the three-dimensional wave equation $(\partial_t^2/v^2 - \nabla^2)\Psi(\rho, \phi, z, t) = 0$ with boundary conditions $\Psi(a, \phi, z, t) = \Psi(\rho, \phi, \pm b, t) = 0$?]

3. Harmonics are the common **multipole** angular solutions of any PDE involving the Laplacian.

a) In the long wavelenth limit $k \to 0$, the Helmholtz equation becomes the Laplace equation $\nabla_\perp^2 \eta = 0$. The eignfunctions of ∂_ϕ are still the *cylindrical harmonics* $\Phi_m(\phi) = e^{im\phi}$, but for the radial solution, $\lim_{k\to 0} J_m(k\rho)$ transforms each Bessel function to its lowest order Taylor approximation. Put the *ansatz* $R_m(\rho) = \rho^\alpha$, into Laplace's equation and solve for α to find two independent solutions $R_m(\rho)$. One which is finite at the origin $\rho = 0$ and the other as $\rho \to \infty$. Show that for

$m=0$, $R(\rho)=\ln(\rho)$ is a second independent solution. Express the *planar harmonics* $R_m(\rho)e^{im\phi}$ which are finite at origin in the form $(x+iy)^m = A_m + iB_m$ and expand A_m, B_m as polynomial solutions to Laplace's equation. Explicitly verify these solutions up to $m=3$.

b) Expand $\nabla^2 V(r,\theta,\phi)$ in spherical coordinates and show the radial part has the forms

$$\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} \;=\; \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} \;=\; \frac{1}{r}\frac{\partial^2}{\partial r^2}r$$

c) Factor out the ϕ-dependence as in #1c) and identify the θ-operator $L^2 = \frac{-d}{dx}(1-x^2)\frac{d}{dx} + \frac{m^2}{1-x^2}$, where $x=\cos\theta$ [distinct from the Cartesian coordinate x and from $x=k\rho$ of #2c)!], to obtain the *general Legendre equation*, for polar waves. Restriction to $m=0$ yields the *Legendre polynomials* of H07#2c. Continuity at the poles $\theta=0,\pi$ requires that $\ell=|m|,|m+1|,|m+2|,\ldots,\infty$. Find all ten polar eigenfunctions $P_\ell^{|m|}(x)$ up to $\ell=3$. Pick two harmonics and show that they are eigenfunctions of L^2. The combined *spherical harmonics* $Y_{lm} = (-1)^m\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_l^m(\cos\theta)e^{im\phi}$ are normalized eigenfunctions of the operator $L^2(\theta,\phi)$ with eigenvalues $\lambda=\ell(\ell+1)$. They represent the atomic s,p,d,f *orbitals* for $\ell=0,1,2,3$. Draw the node lines of each of these ℓ,m-modes on a sphere.

e) Write ∇^2 using the third form of part 3b) and L^2. Factor out the angular dependence and solve the radial Laplace equation for the two eigenfunctions $R_l(r)$ for each ℓ as in part a) to obtain the *solid harmonics* $R_\ell^m(\boldsymbol{r})$ and $I_l^m(\boldsymbol{r})$. Expand $R_\ell^m(\boldsymbol{r})$ in Cartesian coordinates, factoring out the planar harmonic in each. These multinomials in x,y,z are used to label the sub-orbitals.

f) [*bonus*: Solve the Laplace boundary value problem in all space with a point flux source at $\boldsymbol{r}'=(r',\theta',\phi')$ to obtain the potential $V(\boldsymbol{r})=\sum_{lm}R_l^{m*}(\boldsymbol{r}')I_l^m(\boldsymbol{r})$ if $r<r'$ or $\sum_{lm}R_l^{m*}(\boldsymbol{r}')I_l^m(\boldsymbol{r})$ if $r>r'$. $Q_{lm}=\int dq' R_l^{m*}(\boldsymbol{r}')$ is the interior [or $I_l^{m*}(\boldsymbol{r}')$ for the exterior] *multipole moment* of the charge distribution, and $I_l^m(\boldsymbol{r})$ [or $R_l^m(\boldsymbol{r})$] is its corresponding potential. Compare with the point potential Green's function $V(\boldsymbol{r})=\frac{1}{4\pi \mathscr{r}}$ to obtain the *addition theorem* $\frac{1}{\mathscr{r}} = \sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos\gamma)$, where γ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r}'$ and $P_\ell(\cos\gamma)=\frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$, and $r_<$, $r_>$ are the lesser and greater values of r, r', respectively.]

g) [*bonus*: Show that the spherical solution of the Helmholtz equation $(\nabla^2+k^2)j_l(kr)Y_{lm}(\theta,\phi)$ is similar to cylindrical with $m\to\ell+\frac{1}{2}$, and thus the solutions are the *spherical Bessel functions* $j_l(kr)=\sqrt{\frac{\pi}{2kr}}J_{\ell+1/2}(kr)$. The same principle holds for all wave equations in different dimensions. Calculate and illustrate the modes of a spherical wave confined to $r<a$ à la #2d).]