Dipole in dielectric & External field & Free charge $\sigma = \sigma_0 \cos\theta$

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9:17 AM

<u>Note</u>: this practice problem combines <u>FOUR</u> different possible source terms into a single calculation: a) internal dipole b) external [dipole] field, c) dipole free charge distribution, and d) bound charge from a dielectric [a passive "source"]. You can reproduce simpler solutions by setting one or more constants equal to zero.

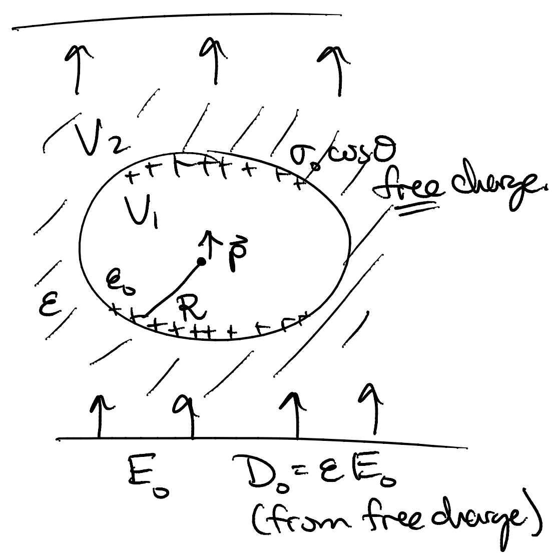
$$V = \sum_{\ell=0}^{\infty} \left(a_\ell r^\ell + b_\ell r^{-\ell-1}\right) P_\ell(\cos\theta)$$

$$V_{\vec{p}} = \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3} = \underbrace{\frac{p}{4\pi\epsilon_0}}_{b_1} \underbrace{r^{-2}\cos\theta}_{r^{-1-1}P_1} \qquad \ell=1$$

$$V_{E_0} = -\int E_0 dz = -E_0 z = \underbrace{-E_0}_{a_1} \underbrace{r^1 \cos\theta}_{r^1 P_1} \qquad \ell=1$$

$$V_1 = \frac{p}{4\pi\epsilon_0} r^{-2} P_1 + \sum_{\ell=0}^{\infty} a_\ell r^\ell P_\ell$$

$$V_2 = -E_0\, r\, P_1 + \sum_{\ell=0}^{\infty} b_\ell r^{-\ell-1} P_\ell$$



* Apply boundary conditions:

$$\Delta V|_b = 0: \quad \left(-E_0 R P_1 + \sum_{\ell=0}^{\infty} b_\ell R^{-\ell-1} P_\ell\right) - \left(\frac{p}{4\pi\epsilon_0} R^{-2} P_1 + \sum_{\ell=0}^{\infty} a_\ell R^\ell P_\ell\right) = 0$$

$$-\Delta \epsilon \frac{\partial V}{\partial n}\Big|_b = 0: \quad -\epsilon_r\left(-E_0 P_1 + \sum_{\ell=0}^{\infty}(-\ell-1) b_\ell R^{-\ell-2} P_\ell\right) + \left(\frac{p}{4\pi\epsilon_0}(-2)R^{-3} P_1 + \sum_{\ell=0}^{\infty} \ell a_\ell R^{\ell-1} P_\ell\right) = \sigma_0/\epsilon_0\, P_1$$

* Separate out components:

if $\ell \neq 1$: $\quad b_\ell R^\ell - a_\ell R^{-\ell-1} = 0$

$$-\epsilon_r(-\ell-1) b_\ell R^{-\ell-2} + \ell a_\ell R^{\ell-1} = 0 \quad \Rightarrow a_\ell = b_\ell = 0$$

(no "source" term)

$\ell = 1$: $\quad -E_0 \cancel{R} + b_1 R^{-3} - \frac{p}{4\pi\epsilon_0} R^{-3} - a_1 \cancel{R} = 0 \qquad$ (divide by R)

$$\epsilon_r E_0 + 2\epsilon_r b_1 R^{-3} - 2\frac{p}{4\pi\epsilon_0} R^{-3} + a_1 = \sigma_0/\epsilon_0$$

b' $\quad p'$ $\quad \sigma'$

$$b' - a_1 = E_0 + p' \qquad (1)$$
$$2\varepsilon_r b' + a_1 = -\varepsilon_r E_0 + 2p' + \sigma' \qquad (2)$$

source terms on right

$$(2) - 2\varepsilon_r (1): \quad (1+2\varepsilon_r)\, a_1 = (-\varepsilon_r - 2\varepsilon_r) E_0 + (2 - 2\varepsilon_r) p' + \sigma'$$
$$(2) + (1): \quad (1+2\varepsilon_r)\, b' = (-\varepsilon_r + 1) E_0 + (2+1) p' + \sigma'$$

$$a_1 = \left(\frac{-3\varepsilon_r}{1+2\varepsilon_r} E_0 + \frac{2-2\varepsilon_r}{1+2\varepsilon_r} \frac{p}{4\pi\varepsilon_0 R^3} + \frac{\sigma_0/\varepsilon_0}{1+2\varepsilon_r} \right)$$

$$b_1' = b_1/R^3 = \left(\frac{1-\varepsilon_r}{1+2\varepsilon_r} E_0 + \frac{3}{1+2\varepsilon_r} \frac{p}{4\pi\varepsilon_0 R^3} + \frac{\sigma_0/\varepsilon_0}{1+2\varepsilon_r} \right)$$

$$V_1(r,\theta) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} + \left(-3\varepsilon_r E_0 + (2-2\varepsilon_r)\frac{p}{4\pi\varepsilon_0 R^3} + \sigma_0/\varepsilon_0 \right) \frac{r\cos\theta}{1+2\varepsilon_r}$$
$$V_2(r,\theta) = -E_0 r\cos\theta + \left((1-\varepsilon_r) E_0 + 3\frac{p}{4\pi\varepsilon_0 R^3} + \sigma_0/\varepsilon_0 \right) \frac{\cos\theta R^3}{(1+2\varepsilon_r) r^2}$$

$$\boxed{V(r,\theta) = -E_0 r\cos\theta + \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} + \frac{(1-\varepsilon_r)E_0 + (2-2\varepsilon_r) + \sigma_0/\varepsilon_0}{1+2\varepsilon_r} \begin{cases} r\cos\theta & r<R \\ \frac{R^3\cos\theta}{r^2} & r>R \end{cases}}$$

$*$ check: i) $V_1 = \frac{R\cos\theta}{1+2\varepsilon_r}\left(\frac{P/\varepsilon_0}{4/3\pi R^3} - 3\varepsilon_r E_0 + \sigma_0/\varepsilon_0 \right) = V_2$ at $r = R$ ✓

ii) $$-\varepsilon_r \frac{\partial V_2}{\partial r}\Big|_R + \frac{\partial V_1}{\partial r}\Big|_R = \Bigg(\varepsilon_r E_0 + \frac{2\varepsilon_r}{1+2\varepsilon_r}\left[(1-\varepsilon_r)E_0 + 3\frac{p}{4\pi\varepsilon_0 R^3} + \sigma_0/\varepsilon_0\right]$$
$$-2\frac{p}{4\pi\varepsilon_0 R^3} + \frac{1}{1+2\varepsilon_r}\left[-3\varepsilon_r E_0 + (2-2\varepsilon_r)\frac{p}{4\pi\varepsilon_0 R^3} + \sigma_0/\varepsilon_0\right]\Bigg)\cdot\cos\theta$$

$$= \left(\varepsilon_r E_0 - 2\frac{p}{4\pi\varepsilon_0 R^3} + \frac{1}{1+2\varepsilon_r}\left[-\varepsilon_r(1+2\varepsilon_r)E_0 + 2(1+2\varepsilon_r)p + (1+2\varepsilon_r)\sigma_0/\varepsilon_0\right]\right)\cos\theta$$

$$= \sigma_0/\varepsilon_0 \cos\theta \quad ✓$$

$*$ limit: $\varepsilon_r \to 1$ $$V = -E_0 r\cos\theta + \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} + \frac{\sigma_0\cos\theta}{3\varepsilon_0}\begin{cases} r & r<R \\ R^3/r^2 & r>R \end{cases}$$