Arizona State University, Physics 311
EXAM 1, 2014-02-17

Instructions: The exam is closed book and timed (50 minutes). Show all steps of calculations. Be careful to pace yourself. Circle alternate problems you wish to be graded. [75 pts maximum]

[15 pts] 1. Given the function $G(r) = 1/4\pi r$, where $r^2 = x^2 + y^2 + z^2$:
a) Calculate the gradient $\nabla G(r)$.
b) Calculate the curl of the gradient $\nabla \times \nabla G(r)$.
c) Calculate the divergence of the gradient $\nabla \cdot \nabla G(r)$.
d) How do $G(r)$, $\nabla G(r)$, and $\nabla \cdot \nabla G(r)$ relate to electrostatics?

3 a) $\nabla G = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{rS_\theta}\frac{\partial}{\partial\phi}\right)\frac{1}{4\pi r} = \frac{-\hat{r}}{4\pi r^2}$

3 b) $\nabla \times \frac{-\hat{r}}{4\pi r^2} = \frac{1}{r^2 S_\theta}\begin{vmatrix} \hat{r} & \frac{1}{r}\hat{\theta} & \frac{1}{rS_\theta}\hat{\phi} \\ \partial_r & \partial_\theta & \partial_\phi \\ \frac{-1}{4\pi r^2} & 0 & 0 \end{vmatrix} = 0$

3 c) $\nabla \cdot \frac{-\hat{r}}{4\pi r^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\hat{r}\left(-\frac{1}{4\pi r^2}\right) + \cdots\cancel{\frac{\partial}{\partial\theta}}\cdots + \cdots\cancel{\frac{\partial}{\partial\phi}}\cdots = 0$

but $\frac{1}{r} \to \infty$ at $r \to 0$

3 so $\int_V \nabla \cdot \frac{-\hat{r}}{4\pi r^2}\,d\tau = \oint_{\partial V} d\vec{a} \cdot \frac{-\hat{r}}{4\pi r^2} = \oint \cancel{r^2}\,d\Omega\,\hat{r} \cdot \frac{-\hat{r}}{4\pi \cancel{r^2}} = \frac{-1}{4\pi}\oint d\Omega = -1$

So $\nabla \cdot \frac{-\hat{r}}{4\pi r^2} = -\delta^3(\vec{r})$

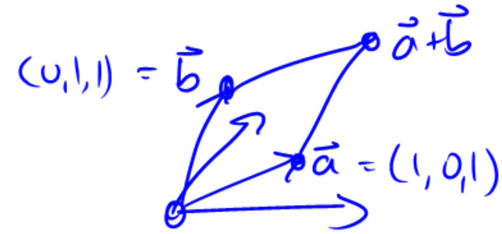
d) $\frac{q}{\varepsilon_0} G(r) =$ potential V of a point charge at the origin

3 $-\frac{q}{\varepsilon_0}\nabla G(r) =$ electric field. $\vec{E}$

$-q\nabla^2 G(r) =$ charge distribution ρ

[20 pts] 2. Calculate the electric field at the point $(0, 0, z)$ on the z-axis due to a flat parallelogram distribution of charge with corners $\mathcal{O} = (0, 0, 0)$, $\boldsymbol{a} = (1, 0, 1)$, $\boldsymbol{b} = (0, 1, 1)$ and $\boldsymbol{a} + \boldsymbol{b} = (1, 1, 2)$ and constant surface charge density σ. You may leave your answer in terms of double integrals containing only constant scalars and parameters of integration.

Alternate problem [10 pts]: Calculate the electric field $\boldsymbol{E}(0, 0, z)$ due to a ring of linear charge density λ and radius a, in the xy-plane, centered about the origin.



let $\vec r' = \vec a\,\alpha + \vec b\,\beta$

5 $d\vec l' = d\vec r' = \underbrace{\vec a\, d\alpha}_{dl_\alpha} + \underbrace{\vec b\, d\beta}_{dl_\beta}$

$d\vec a' = \vec a\, d\alpha \times \vec b\, d\beta = \begin{vmatrix} \hat x & \hat y & \hat z \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} d\alpha\, d\beta = (-\hat x - \hat y + \hat z)\, d\alpha\, d\beta$

5

$da' = |d\vec a| = \sqrt{3}\, d\alpha\, d\beta$

5 $\vec{\mathfrak{r}} = \hat z z - \vec a \alpha - \vec b \beta = \hat x(-\alpha) + \hat y(-\beta) + \hat z(z - \alpha - \beta)$

$\mathfrak{r}^2 = \alpha^2 + \beta^2 + (z - \alpha - \beta)^2$

$$\vec E(0,0,z) = \int_{\alpha=0}^{1}\int_{\beta=0}^{1} \frac{\sigma \cdot \sqrt{3}\, d\alpha\, d\beta\, \left(\hat x(-\alpha) + \hat y(-\beta) + \hat z(z - \alpha - \beta)\right)}{4\pi\varepsilon_0 \left(\alpha^2 + \beta^2 + (z - \alpha - \beta)^2\right)^{3/2}}$$

5

$$= \frac{\sqrt{3}\sigma}{4\pi\varepsilon_0}\left[z\hat z I_0 - (\hat x + \hat y + 2\hat z) I_1\right] \qquad I_0 = \int_{\alpha=0}^{1}\int_{\beta=0}^{1} \frac{d\alpha\, d\beta}{\left(\alpha^2 + \beta^2 + (z - \alpha - \beta)^2\right)^{3/2}}$$

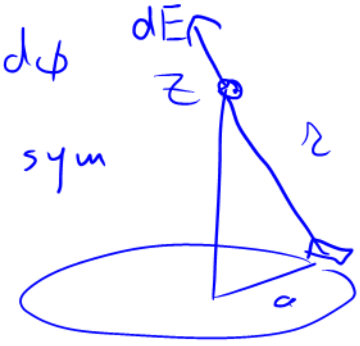
$$I_1 = \int_{\alpha=0}^{1}\int_{\beta=0}^{1} \frac{\alpha\, d\alpha\, d\beta}{\left(\alpha^2 + \beta^2 + (z - \alpha - \beta)^2\right)^{3/2}}$$

Alternate:

3 $\vec r'(\phi) = \hat s a = \hat x a c_\phi + \hat y a s_\phi$

3 $\vec r = \hat z z \qquad \vec{\mathfrak{r}} = \hat z z - \hat s a \qquad dq' = \lambda a\, d\phi$

$\mathfrak{r}^2 = z^2 + a^2$



4 $$\vec E = \int \frac{dq'\, \vec{\mathfrak{r}}}{4\pi\epsilon_0 \mathfrak{r}^3} = \int_{\phi=0}^{2\pi} \frac{dq'\,(\hat z z - \overbrace{\cancel{\hat s a}}^{0 \text{ by sym}})}{4\pi\varepsilon_0 (z^2 + a^2)^{3/2}}$$

$$= \frac{\lambda \cdot 2\pi a z \hat z}{4\pi\varepsilon_0 (z^2 + a^2)^{3/2}}$$

[20 pts]3. Derive the Helmholtz theorem from the vector identity $\nabla^2 \boldsymbol{F} = \nabla\nabla\cdot\boldsymbol{F} - \nabla\times\nabla\times\boldsymbol{F}$, and use it to derive Coulomb's law $\boldsymbol{E}(\boldsymbol{r}) = \int_{\mathcal{V}'} \rho(\boldsymbol{r}')d\tau'\,\hat{\boldsymbol{\imath}}\,/4\pi\epsilon_0 \imath^2$, given only the divergence $\nabla\cdot\boldsymbol{E} = \rho/\epsilon_0$ and curl $\nabla\times\boldsymbol{E} = 0$ of the electric field $\boldsymbol{E}$.

Alternate problem [12 pts]: Prove that $\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{0}$ ensures that one can well-define [independent of path of integration] a potential $V(\boldsymbol{r}) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{E}\cdot\boldsymbol{dl}$. Show that the electric field is the negative gradient of this potential, $\boldsymbol{E} = -\boldsymbol{\nabla}V$. Derive the formula $V(\boldsymbol{r}) = \int_{\mathcal{V}'} \rho(\boldsymbol{r}')d\tau'/4\pi\epsilon_0 \imath$ from Coulomb's law.

a) $\vec{F} = -\nabla\underbrace{(-\nabla^{-2}\underbrace{\nabla\cdot\vec{F}}_{\rho/\epsilon})}_{V} + \nabla\times\underbrace{(-\nabla^{-2}\underbrace{\nabla\times\vec{F}}_{\vec{J}})}_{\vec{A}}$

6

since $V = -\nabla^{-2}\rho \qquad \rho/\epsilon_0 = -\nabla^2 V$

let $\rho(\vec{r}) = \int d\tau'\,\rho(\vec{r}')\,\delta^3(\vec{r}-\vec{r}')$

6 then $-\nabla^{-2}\rho(\vec{r}) = \int d\tau'\,\rho(\vec{r}')\left(-\nabla^{-2}\delta^3(\vec{\imath})\right)$

$= \int d\tau' \frac{\rho(\vec{r}')}{4\pi\imath}$

so $\vec{F} = -\nabla\int d\tau' \frac{\nabla'\cdot F'}{4\pi\imath} + \nabla\times\int d\tau' \frac{\nabla'\times F'}{4\pi\imath}$

8 put in $\vec{F}\to\vec{E} \quad \nabla\cdot\vec{E} = \rho/\epsilon_0 \quad \nabla\times\vec{E} = \vec{0}$

$\vec{E} = -\nabla\int d\tau' \frac{\rho(\vec{r}')}{4\pi\epsilon_0\imath} + \cancel{\nabla\times\int d\tau' \frac{\vec{0}}{4\pi\imath}}$

$= \int d\tau' \frac{\rho(\vec{r}')}{4\pi\epsilon_0}\left(-\nabla_\imath \frac{1}{\imath}\right) = \int d\tau' \frac{\rho(\vec{r}')\,\hat{\imath}}{4\pi\epsilon_0\imath^2}$

b) Since $\nabla\times E = 0$, $\oint_{\partial S} \vec{E}\cdot d\vec{l} = \int_S \nabla\times E\cdot d\vec{a} = 0$

4 so $\int_{P_1\,a}^{b} \vec{E}\cdot d\vec{l} + \int_{P_2\,b}^{a} \vec{E}\cdot d\vec{l} = 0 \qquad \int_{P_1\,a}^{b} \vec{E}\cdot d\vec{l} = \int_{P_2\,a}^{b} \vec{E}\cdot d\vec{l}$

4 By the FTVC, $-\int_a^b \nabla V\cdot d\vec{l} = -V(b) + V(a) = \int \vec{E}\cdot d\vec{l}$

so $\vec{E} = -\nabla V$.

$d\vec{l} = \hat{\imath}\,d\imath + \dots$

4 $V(\vec{r}) = -\int_\infty^r \int d\tau' \frac{\rho(\vec{r}')\,\hat{\imath}\cdot d\vec{l}}{4\pi\epsilon_0\imath^2} = \int d\tau'\,\rho(\vec{r}') \int_{r'=\infty}^{r} \frac{-d\imath}{\imath^2} = \int d\tau' \frac{\rho(r')}{4\pi\epsilon_0\imath}$

[20 pts] 4. Essay question (paragraph form): explain in detail the geometry of electrostatics. You may refer to illustrations, but the problem will be graded on the content and organization of the written response.

Examples to get started: describe the geometric representations flux $\Phi_E = \int \boldsymbol{E} \cdot d\boldsymbol{a}$ and flow $\mathcal{E}_E = \int \boldsymbol{E} \cdot d\boldsymbol{l}$ of the electric field $\boldsymbol{E}(\boldsymbol{r})$, and the relationship between each of them. How do they relate to charge, field lines, and equipotentials? Explain the integral and differential field equations in terms of these geometric concepts. What about projections? Why use fields at all? Why would one expect a $1/r^2$ force law?

20

A field has two complementary represtations as either field (flux) lines or equipotential (flow) surfaces. The field is the tangent of the field lines and gradient of the potential. Thus the field lies and equipotentials are perpendicular. The field lines count the units of flux through a surface, while the equipotentals count the amount of flow along a path. In each case, count the # of intersections between curves & surfces. The magnitude of the field equals either the flux density or density of equipotentials. The source of flux is divergence: every field line starts at a point, which represents a unit of divergence or charge, for electric fields. You can either count the number of charges in a volume or the total number of field lines leaving the volume (Gauss' law). The flow of the electric field represents change in potential [energy/charge]. Since the electric field is conservative, potential is well-defined everywhere and flow is just the change in potential. Every equipotential surface is closed, so that the change in potential is independent of the path. Otherwise, an "edge" of an equipotential would represent a unit of curl; Thus the curl of $\vec{E}$ is 0. You can count the lines of curl through a surface, or the number of equipotentials passed while passing around the perimeter of the surface.