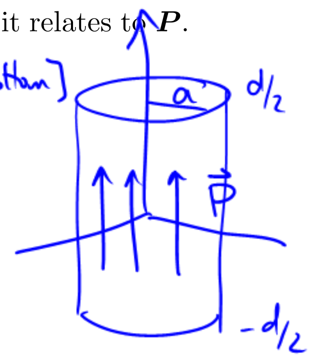
Arizona State University, Physics 311
EXAM 2, 2014-03-26

Instructions: The exam is closed book and timed. Show all intermediate steps. Pace yourself. Save integrals until the end. Circle alternate problems you wish to be graded. [55 pts maximum]

[10 pts] 1. A cylindrical rod of height d and radius a from the z-axis is centered on the origin and has polarization $\boldsymbol{P} = P_0\hat{\boldsymbol{z}}$.
a) Calculate the bound charge density ρ_b inside the rod and σ_b on the surface of the rod.
b) Calculate the total dipole moment $\boldsymbol{p}$ of the bound charge and explain how it relates to $\boldsymbol{P}$.



5 a) $\rho_b = \nabla\cdot P = \partial_z P_0 = 0$ $\quad \sigma_b = \hat{n}\cdot\vec{P} = +P_0$ [top] $-P_0$ [bottom]
$= 0$ [sides]

5 b) $\vec{p} = \int dq'\,\vec{r}\,' = \int_{top} P_0\, da\,[\cancel{x},\cancel{y},d/2] + \int_{bot} -P_0\, da\,[\cancel{x},\cancel{y},-d/2]$

$= P_0\,\pi a^2\cdot d/2\,\hat{z} - -P_0\,\pi a^2 d/2\,\hat{z} = P_0\,\pi a^2 d\,\hat{z}$

$= \vec{P}\cdot$ volume.

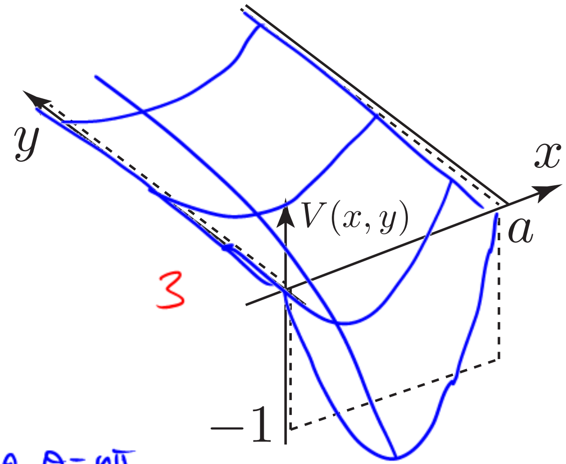
[10 pts] 2. Essay question: explain the meaning and purpose of all three vector fields in the equation $\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P}$. How are each similar and different? Explain the physical/geometric significance of the flux and source of each field and how they relate to each other. Use these concepts to explain what happens when a dielectric is placed in an electric field.

10 $\epsilon\vec{E}$ is the field which represents the force on a test particle. lines of flux diverge from any charge, bound or free.

If a material is polarizable $\vec{E}$ will polarize individual molecules into dipoles, $\vec{p}$ and the dipole density is $\vec{P}$, similar to ρ being a charge density. One may line dipoles end on end to form $\vec{P}$-chains, which are lines of flux of the P-field. There will be an uncancelled bound charge at each end. Thus the "source" of $\vec{P}$ is ρ_b (or σ_b at the boundary) even though it is "created" by $\vec{E}$.

Thus every uncancelled negative bound charge terminates an ϵE-line of flux and starts a P-chain. You can think of $\epsilon_0 E$, P as being a single flux line going back and forth between ϵE and P. We call these D-flux lines, and these only start or end at free charges.

[15 pts] 3. Calculate the first non-zero term (of the general solution) for the the electric potential $V(x, y)$ in the region $0 < x < a$ and $0 < y < \infty$, with the boundary conditions $V(0, y) = V(a, y) = 0$ and $V(x, 0) = -1$. Plot this solution.



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$X(x) = \cos(kx)$ or $\sin(kx)$

i) use $\sin(kx)$ since $\sin(0) = 0$.

$Y(y) = e^{\pm ky}$

ii) $V(x=a) = Y(y) \cdot \sin(ka) = 0$ $\quad \sin(\theta) = 0$ @ $\theta = n\pi$

$k_n = \frac{n\pi}{a}$

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iii) $V(y \to \infty) = 0 \to$ use e^{-ky} so it doesn't blow up.

Thus $V = \sum_{n=1}^{\infty} A_n e^{-k_n y} \sin k_n x$

iv) $V(x,0) = -1 = \sum_{n=1}^{\infty} A_n e^{-k_n \cdot 0} \sin k_n x$

$$\int_0^a \sin k_1 x \cdot -1 \, dx = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^a \sin(k_1 x) \cdot \sin(k_n x)}_{0 \text{ if } n \neq 1 \quad a/2 \text{ if } n=1} = A_1 \cdot a/2$$

6

$$-\int_0^a \sin \frac{\pi x}{a} dx = + \frac{a}{\pi} \cos \frac{\pi x}{a} \Big|_0^a = \frac{a}{\pi}(\cos \pi - \cos 0) = \frac{-2a}{\pi} \qquad A_1 = \frac{-4}{\pi}$$

$$V(x,y) = \frac{-4}{\pi} e^{-\frac{\pi y}{a}} \sin\left(\frac{\pi x}{a}\right) + \dots \frac{3\pi}{a} \dots$$

[20 pts] 4. Calculate the electric potential inside and outside a dielectric sphere of electric susceptibility χ and radius R centered about the origin in a constant external electric field $\boldsymbol{E}_0 = E_0\hat{z}$. What is the total dipole moment of the sphere? Draw the $\boldsymbol{\epsilon}_0\boldsymbol{E}$, $\boldsymbol{P}$, and $\boldsymbol{D}$ field lines and the equipotentials.

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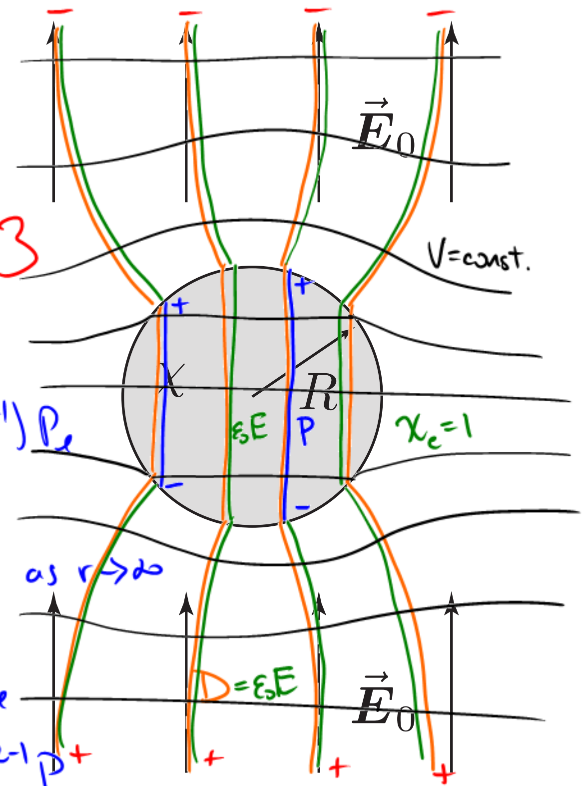


Alternate problem [10 pts]: Calculate $V(r,\theta)$ inside and outside a spherical (surface) charge distribution $\sigma(R,\theta) = \sigma_0\cos\theta$.

$V_1 = \sum_l \left(A_l r^l + B_l r^{-l-1}\right) P_l \qquad V_2 = \sum_l \left(C_l r^l + D_l r^{-l-1}\right) P_l$

1 i) $B_l = 0$ V finite at $r \to 0$

3 iv) $C_l = -E_0$ $(l=1)$ or 0 $(l \neq 1)$ $\quad V \to -E_0 r\cos\theta$ as $r \to \infty$

$V_1 = \sum_l A_l r^l P_l \qquad V_2 = -E_0 r P_1 + \sum_l D_l r^{-l-1} P_l$

2 ii) $V_1(R) = V_2(R)$: $\sum_l A_l R^l P_l = -E_0 R P_1 + \sum_l D_l R^{-l-1} P_l$

3 iii) $-\epsilon_0 \frac{\partial V_2}{\partial r}(R) + \epsilon \frac{\partial V_1}{\partial r}(R) = 0$: $-\epsilon_0\left(-E_0 P_1 + \sum_l (-l-1) D_l R^{-l-2} P_l\right) + \epsilon\left(\sum_l (l) A_l R^{l-1}\right) = 0$

2 if $l \neq 1$ $\quad A_l = D_l = 0$

if $l = 1$ $\quad A_1 + E_0 = D_1/R^3 \qquad \epsilon_0 E_0 + 2D_2/R^3 + \epsilon A_1 = 0$

$\epsilon_0 E_0 + 2\epsilon_0(A_1 + E_0) + \epsilon A_1 = 0$

$3\epsilon_0 E_0 + (2\epsilon_0 + \epsilon) A_1 = 0$

4 $A_1 = \frac{-3\epsilon_0}{\epsilon + 2\epsilon_0} E_0 = \frac{-3}{3+\chi} E_0 \qquad D_1 = (A_1 + E_0) R^3 = \frac{\chi}{3+\chi} R^3 E_0$

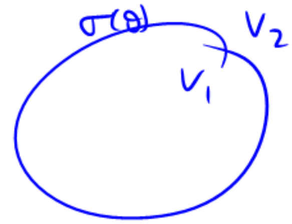
$V_1 = \frac{-3}{3+\chi} E_0 z \qquad V_2 = -E_0 z + \frac{\chi}{3+\chi} R^3 E_0 \frac{\cos\theta}{r^2} = -E_0 z + \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$

so $\vec{p} = \frac{4}{3}\pi R^3 \cdot \frac{3\chi}{3+\chi} \epsilon_0 \vec{E}_0 = \tau \cdot \vec{P}_{in}$

check: on the inside,
$\vec{P} = \chi\epsilon_0\vec{E} = \frac{3\chi}{3+\chi}\epsilon_0\vec{E}$ ✓ 2

Alternate:

2 $V_1 = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \cancel{B_\ell}^{\,0} r^{-\ell-1}\right) P_\ell \qquad V_2 = \sum_{\ell=0}^{\infty}\left(\cancel{C_\ell}^{\,0} r^\ell + D_\ell r^{-\ell-1}\right) P_\ell$



$V_1(R)=V_2(R): \quad \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell = \sum_{\ell=0}^{\infty} D_\ell R^{-\ell-1} P_\ell$

3 $D_\ell = A_\ell R^{2\ell+1}$

$-\epsilon_2 \frac{\partial V_2}{\partial r} + \epsilon_1 \frac{\partial V_1}{\partial r} = \sigma: \quad -\epsilon_0 \sum_{\ell=0}^{\infty} D_\ell (-\ell-1) R^{-\ell-2} P_\ell + \epsilon_0 \sum_{\ell=0}^{\infty} A_\ell (\ell) R^{\ell-1} P_\ell = \sigma$

3 $\sum_{\ell=0}^{\infty} \epsilon_0 (2\ell+1) R^{\ell-1} A_\ell P_\ell(\cos\theta) = \sigma(\theta) = \sigma_0 \cos\theta.$

so $A_\ell = 0$ if $\ell \neq 1 \qquad \epsilon_0 \cdot 3 R^0 A_1 = \sigma_0$

2 $V_1 = \frac{\sigma_0}{3\epsilon_0} r\cos\theta \qquad V_2 = \frac{\sigma_0}{3\epsilon_0} \frac{R^3 \cos\theta}{r^2}$