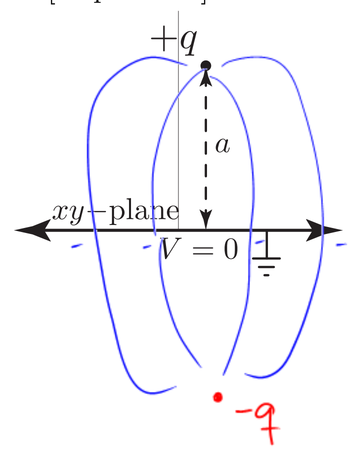
University of Kentucky, Physics 416G
EXAM 3, 2012-11-09

Instructions: The exam is closed book and timed (50 minutes). Show all steps of calculations. Be careful to pace yourself; you may want to set up all integrals before evaluation. [40 pts total]

1. A point charge q is placed a distance a from an infinite grounded conducting plane at $z = 0$.



[6 pts] a) Calculate $\vec{E}(x, y, z)$ everywhere above the plane.

$$\vec{E}(x,y,z) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\hat{s}s + \hat{z}(z-a)}{(s^2+(z-a)^2)^{3/2}} - \frac{\hat{s}s + \hat{z}(z+a)}{(s^2+(z+a)^2)^{3/2}}\right]$$

$$\hat{s}s = \hat{x}x + \hat{y}y$$

$$s^2 = x^2+y^2$$

[8 pts] b) Calculate the induced surface charge $\sigma(x, y)$ on the plane AND the total induced charge on the plane.

$$\sigma(x,y) = \varepsilon_0 E_z\Big|_{x=0,\,y=0} = \frac{q}{4\pi}\left(\frac{-a}{(s^2+a^2)^{3/2}} - \frac{a}{(s^2+a^2)^{3/2}}\right) = \frac{-2qa}{4\pi(s^2+a^2)^{3/2}}$$

$$q_{ind} = \int_{s=0}^{\infty}\int_{\phi=0}^{2\pi} \sigma \underbrace{da}_{s\,ds\,d\phi} = \int_{\phi=0}^{2\pi} d\phi \int_{s=0}^{\infty} \frac{-2qa\,s\,ds}{4\pi\,(s^2+a^2)^{3/2}}$$

let $u = s^2+a^2$
$du = 2s\,ds$

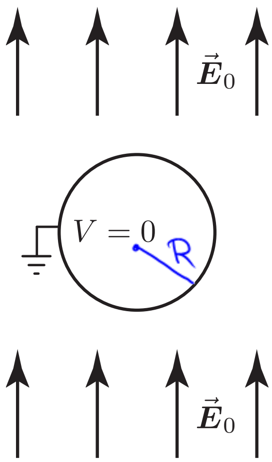
$$= \frac{-2\pi\, qa}{4\pi}\int_{u=a^2}^{\infty} u^{-3/2}\,du = -\frac{qa}{2}\,\frac{u^{-1/2}}{-1/2}\Big|_{a^2}^{\infty} = -\frac{qa}{2}\left(0 - \frac{(a^2)^{-1/2}}{-1/2}\right) = -q$$

[6 pts] c) Calculate the three lowest moments of this charge distribution (real + image charge):
$Q^{(0)} = \int \bar{d}q'$, $Q^{(1)} = \int \bar{d}q' z'$, $Q^{(2)} = \int \bar{d}q' \frac{1}{2}(3z'z' - r'^2)$. Show all work.

$Q^{(0)} = \Sigma q_i = q - q = 0$ $\quad Q^{(1)} = \Sigma q_i z_i = qa + (-q)(-a) = 2qa \; (= p)$

$Q^{(2)} = \Sigma q_i \frac{1}{2}(3z'^2 - r'^2) = q\cdot\frac{1}{2}(3a^2 - a^2) + -q\cdot\frac{1}{2}(3a^2-a^2) = 0$

2. [10 pts] a) A grounded sphere is placed in an external electric field $\vec{\boldsymbol{E}} = E_0\hat{\boldsymbol{z}}$. Starting from the general solution of $\nabla^2 V = 0$ in spherical coordinates: $V(r,\theta) = \sum_{\ell=0}^{\infty}(A_\ell r^\ell + B_\ell r^{-\ell-1)})P_\ell(\cos\theta)$, calculate the electric potential everwhere outside the sphere.



outside, $V_{ext} = -E_0 z$ so that $E = -\nabla V = E_0\hat{z}$

$= -E_0 r^1 P_1(\cos\theta)$ so $A_1 = -E_0$ $A_n = 0$ otherwise.

$$V(R,\theta) = 0 = -E_0 r^1 P_1(\cos\theta) + \sum_{\ell=0}^{\infty} B_\ell r^{-\ell-1} P_\ell(\cos\theta) \Big|_{r=R}$$

$B_\ell = 0$ for $\ell \neq 1$

$-E_0 R + B_1 R^{-2} = 0$ $\quad B_1 = E_0 R^3$

$$V(r,\theta) = E_0\left(-r + \frac{R^3}{r^2}\right)\cos\theta.$$

[5 pts] b) Calculate surface charge density on the sphere, $\sigma(\theta)$.

$$\sigma/\epsilon_0 = -\frac{\partial V}{\partial r} = E_0\left(1 - (-2)\frac{R^3}{R^3}\right)\cos\theta = 3E_0\cos\theta.$$

[5 pts] c) Evaluate the dipole moment of the charge on the sphere. Explain how your answer is related to the general solution above. If you can't solve part b), use the charge distribution $\sigma(\theta) = \sigma_0\cos\theta$.

ie $\sigma_0 = 3\epsilon_0 E_0$

$$p_z = \int dq' z' = \int_{x=-1}^{1} \underbrace{\sigma_0\cos\theta'}_{\sigma} \cdot \underbrace{dx\, 2\pi R^2}_{da} \underbrace{Rx}_{z'} = 2\pi R^3\sigma_0 \int_{-1}^{1} x^2\, dx = 2\pi R^3\sigma_0 \underbrace{\frac{x^3}{3}\Big|_{-1}^{1}}_{2/3}$$

$x = \cos\theta'$

$$= \frac{4\pi}{3}R^3\sigma_0 = 4\pi R^3\epsilon_0 E_0$$

note: $q_0 = \int_0^1 \sigma_0\cos\theta'\, dx\, 2\pi R^2 = \sigma_0 \cdot 2\pi R^2 \underbrace{\int_0^1 x\, dx}_{1/2} = \sigma_0 \pi R^2$

so $p_z = \frac{4}{3} q_0 R$