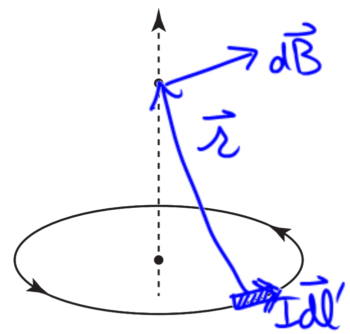
Arizona State University, Physics 311
EXAM 3, 2014-04-21

Instructions: The exam is closed book and timed. Show intermediate steps. [45 pts]

[8 pts] 1. Calculate the magnetic field along z-axis due to a circular loop of wire of radius a centered on the xy-plane with current $+I$ in the $\hat{\boldsymbol{\phi}}$ direction.



4

$$\vec{r}' = \hat{s}_{\phi'} a = \hat{x}\, a\cos\phi' + \hat{y}\, a\sin\phi'$$

$$d\vec{l}' = \hat{\phi}'\, a\, d\phi' = -\hat{x}\cdot a\cdot\sin\phi' + \hat{y}\, a\cdot\cos\phi'$$

$$\vec{\mathscr{r}} = z\hat{z} - \hat{s}_{\phi'} a = z\hat{z} - \hat{x}\cdot a\cdot\cos\phi' - \hat{y}\cdot a\cdot\sin\phi'$$

$$\mathscr{r}^2 = z^2 + a^2$$

4

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}'\times\vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{\mu_0 I}{4\pi}\int_{\phi=0}^{2\pi} \frac{\hat{\phi}' a\, d\phi' \times (z\hat{z} - \hat{s}_{\phi'} a)}{(z^2+a^2)^{3/2}}$$

$\hat{\phi}_{\phi'} \times \hat{z} = \hat{s}_{\phi'}$ integrates to zero.
$\hat{\phi}_{\phi'} \times \hat{s}_{\phi'} = -\hat{z}$

$$= \frac{\mu_0 I a^2 \hat{z}}{4\pi(z^2+a^2)^{3/2}}\int_0^{2\pi} d\phi = \frac{\mu_0 I a^2}{2(z^2+a^2)^{3/2}}\hat{z} = \frac{\mu_0 \vec{m}}{2\pi(z^2+a^2)^{3/2}}$$

[10 pts] 2. Show that the magnetostatic field equations $\nabla\cdot\boldsymbol{B} = 0$ and $\nabla\times\boldsymbol{B} = \mu\boldsymbol{J}$ ensure the existance of a vector potential $\boldsymbol{A}$. Is the potential unique, or how can it transform and still be valid? Show that $-\nabla^2\boldsymbol{A} = \mu\boldsymbol{J}$, assuming $\nabla\cdot\boldsymbol{A} = 0$.

3

since $\nabla\cdot\vec{B} = 0$, there exists a vector field $\vec{A}(\vec{r})$ such that $B = \nabla\times\vec{A}$ by inv. Poincaré (potential) theorem.

if $\nabla\times A_1 = \nabla\times A_2 = \vec{B}$ then $\nabla\times(A_1 - A_2) = 0$ and $A_1 - A_2 = \nabla\chi$ for some scalar gauge $\chi(\vec{r})$

3

by the Helmholtz theorem, a field $\vec{A}$ is uniqely determined by its div & curl, and $\nabla\times\vec{A} = \vec{B}$, thus we can fix the gauge by specifying $\nabla\cdot\vec{A} = 0$ (Coulomb gauge)

4

then $\nabla\times\vec{B} = \nabla\times(\nabla\times\vec{A}) = -\nabla^2\vec{A} + \nabla\cancel{\nabla\cdot\vec{A}} = \mu\vec{J}$

$$-\nabla^2\vec{A} = \mu\vec{J} \quad \text{[Poisson Eq]}$$

[12 pts] 3. Essay question: compare and contrast electrostatics and magnetostatics, both in terms of phenomenology [observations], and conceptual formalism. Be sure to include electic and magnetic materials, and the flux and flow of relevant vector fields in your discussion.

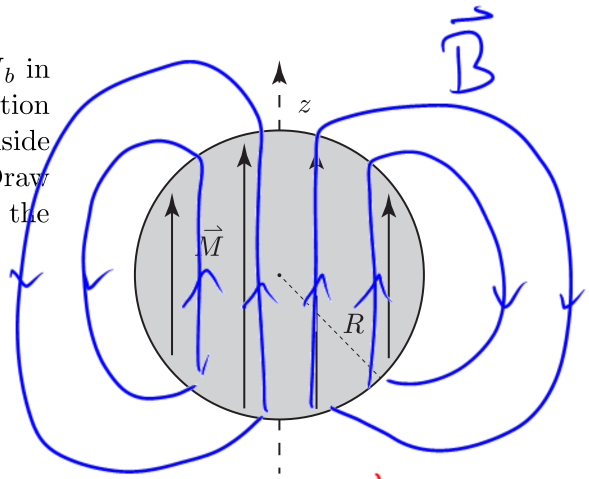
There is a mixed symmetry between electric and magnetic fields –
they follow almost the same formalism, but with a reversal between the
source / force role of the flux / flow [longitudinal / transverse] parts
2 of the field. The net effect is substantial: with divergence and gradients
being replaced by curls, a vector potential instead of scaler, and a
vector Poisson's equation. The cause for this traces back to the
difference between charge and current, the source of electric & magnetic
2 fields respectively. Charge dq is a density, while the current element
$d\vec{q} = \rho\vec{v} = \vec{J}d\tau$ is a differential vector field. Thus while electric field
lines ($\vec{D}$) eminate from charges, it is magnetic flow sheets ($\vec{H}$) that
emanate from current-carrying wires. This means magnetic fields are
2 generated by "sources" of curl, not divergence, and the appropriate
integral equation is Ampère's law $\mathcal{E}_H = I$ not Gauss' law. In contrast
there is no magnetic "charge" monopoles, and B-field lines are continuous.
This absence of divergence gives rise to a vector potential, which
2 now has gauge dependence $\vec{A} \to \vec{A} + \nabla\chi$ (or symmetry) as opposed to a
simple constant offset (ground: $V=0$). There is an addition equation
representing continuity of charge ($\nabla\cdot\vec{J} = 0$).

In the absence of sources, the equations are identical and
admit a magnetic scalar potential, although its physical interpretation
is in terms of bounding currents, not energy. The vector potential
$\vec{A}$ is the dynamical one, representing momentum of the field.

2 Dipoles can be represented by N and S poles in alalogy with electrostatics,
although the field goes the opposite direction inside the dipole. Thus,
magnetization can be aligned to form chains (or M-solenoids). These do
not shield the B-field like polarization chains, but actually carry a
B-field line inside thus B-field lines are continuous, and alternate
between M-solenoids and H-lines at each N or S pole.

2 Current loops can also mesh side-by-side to form an M-mesh
or flow-sheet of $\vec{M}$, bounded by the net bound current, just as P-chains
are bounded by bound charge. The M-sheets add to the H-sheets to form
the total magnetic field $\vec{B} = \mu_0(\vec{H} + \vec{M})$, as above. Thus $\vec{B}$ tends to be magnified
by materials.

[15 pts] 4. Calculate the bound current densities $\boldsymbol{K}_b$ and $\boldsymbol{J}_b$ in and around a sphere of radius R with constant magnetization $\boldsymbol{M} = M_0 \hat{\boldsymbol{z}}$. Calculate the magnetic scalar potential U inside and outside and the magnetic field $\boldsymbol{B}$ inside the sphere. Draw the magnetic flux lines. How does you solution differ from the electric case with constant polarization $\boldsymbol{P} = P_0 \hat{\boldsymbol{z}}$?



1

3 a) $\vec{K}_b = -\hat{n} \times \vec{M} = -\hat{r} \times M_0 \hat{z} = +M_0 \sin\theta \, \hat{\phi}$

$\vec{J}_b = \nabla \times \vec{M} = \nabla \times M_0 \hat{z} = \vec{0}$

2 b) $U_1 = \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-l-1}) P_l(\cos\theta)$

$U_2 = \sum_{l=0}^{\infty} (C_l r^l + D_l r^{-l-1}) P_l(\cos\theta)$

2 B.C.: i) as $r \to 0$ U_1 finite $\to B_l = 0$

ii) as $r \to \infty$ U_2 finite $\to C_l = 0$

iii) $-\dfrac{\partial U_2}{R \partial\theta} + \dfrac{\partial U_1}{R\partial\theta} = K_\phi$

2 $\sum_{l=0}^{\infty} [-D_l R^{-l-2} + A_l R^{l-1}] P_l'(\cos\theta) \cdot (-\sin\theta) = M_0 \sin\theta$

$-\dfrac{\partial U_2}{\partial r} + \dfrac{\partial U_1}{\partial r} = 0$

iv) $\sum_{l=0}^{\infty} \mu_0 [-(-l-1) D_l R^{-l-2} + l A_l R^{l-1}] P_l(\cos\theta) = 0$

2 $A_l = D_l = 0$ for all $l \neq 1$, and $P_1'(x) = 1$ $P_1(x) = x$

thus

iii) $-D_1/R^3 + A_1 = -M_0$

2 iv) $2D_1/R^3 + A_1 = 0$

$2 \cdot$(iii) + (iv): $3A_1 = -2M_0$

(iii) − (iv): $-3D_1/R^3 = -M_0$

$U_1 = -\frac{2}{3} M_0 r \cos\theta = -\frac{2}{3} M_0 z$

$U_2 = \frac{1}{3} M_0 \frac{R^3}{r^2} \cos\theta$

$\vec{B} = -\mu_0 \nabla H_1 = \frac{2}{3} \mu_0 \vec{M}$

(since treating $\vec{M}$ as effective currents)

1 For the polarized sphere, the electric field inside is half as strong, in the opposite direction. (ie, the same as the $\vec{H}$-field!)