

Example 4.7

Wednesday, March 19, 2014
10:33 AM

$$V = \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-l-1}) P_l(\cos \theta)$$

sph

← Griffiths 4.7

$$V = \sum_{m=1}^{\infty} (A_m s^m + B_m s^{-m}) (C_m \sin(m\phi) + D_m \cos(m\phi))$$

cyl

← HW 7 #1

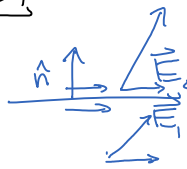
Bulk Differential Equations

$$\epsilon = \epsilon_0 \epsilon_r = \epsilon_0 (1 + \chi_e)$$

$$\begin{aligned} \boxed{\begin{aligned} \nabla \cdot \vec{E} &= \rho_f \\ \nabla \cdot \vec{D} &= \rho_f \end{aligned}} \Rightarrow \vec{E} = -\nabla V \quad \vec{D} = -\epsilon \nabla V \\ \boxed{\nabla \cdot \vec{D} = \rho_f} \Rightarrow \nabla \cdot \epsilon \nabla V = \rho_f = 0 \\ \boxed{\nabla^2 V = 0} \end{aligned}$$

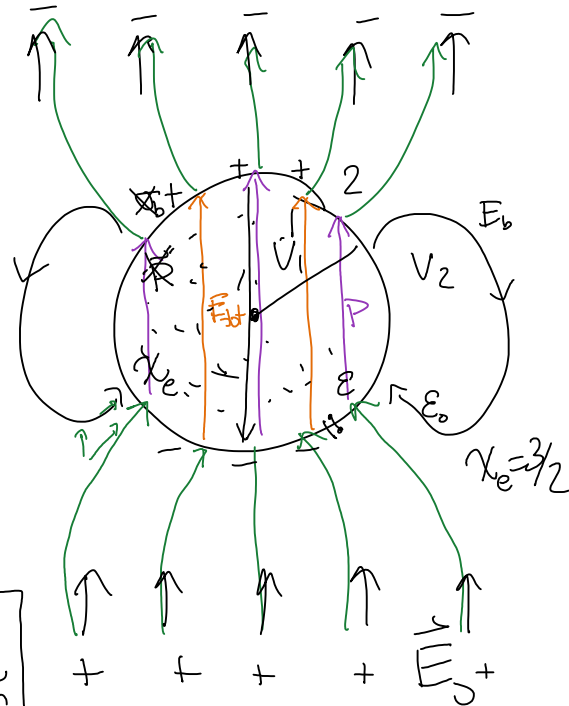
Boundary Conditions

$$\nabla \rightarrow \Delta \hat{n} \quad \tau \rightarrow a$$



$$\begin{aligned} \Delta \hat{n} \times \vec{E} &= 0 & E_{2t} - E_{1t} &= 0 \\ \Delta \hat{n} \cdot \vec{D} &= \sigma_f & D_{2n} - D_{1n} &= \sigma_f \end{aligned}$$

$$\boxed{\begin{aligned} V_1 &= V_2 \\ -\epsilon_2 \frac{\partial V_2}{\partial n} + \epsilon_1 \frac{\partial V_1}{\partial n} &= \sigma_f \end{aligned}}$$



$$\begin{aligned} \hat{n} \cdot (\vec{D} - \epsilon \vec{E} = \epsilon (-\nabla V)) &= -\epsilon \frac{\partial V}{\partial n} \\ \hat{n} \cdot \vec{D} &= (\hat{n} \cdot \nabla) \dots = \frac{\partial}{\partial n} \end{aligned}$$

General Solutions

$$V_1 = \sum_l (a_l r^l + b_l r^{-l-1}) P_l$$

$$V_2 = \sum_l (c_l r^l + d_l r^{-l-1}) P_l$$

Application of Boundary Conditions

i) $V(r \rightarrow 0)$ finite: $b_l = 0$ for all l .

ii) $V(r \rightarrow \infty)$ blows up properly $\rightarrow -E_0 z \quad \nabla V = E_0 \hat{z}$
 $V \rightarrow -E_0 r^1 P_1(\cos\theta) \quad l=1$
 $\Rightarrow C_1 = -E_0 \quad C_0 = C_2 = C_3 = \dots = 0.$

$$V_1 = \sum_l a_l r^l P_l \quad V_2 = -E_0 r P_1 + \sum_l d_l r^{-l-1} P_l$$

iii) $V_1(R) = V_2(R): \sum_l a_l R^l \underbrace{P_l}_{\text{basis}} = -E_0 R \underbrace{P_1}_{\text{basis}} + \sum_l d_l R^{-l-1} \underbrace{P_l}_{\text{basis}}$

$$l \neq 1: \boxed{a_l R^l = d_l R^{-l-1}}$$

$$l=1: a_1 R = -E_0 R + d_1 R^{-2} \quad d_1/R^3 = a_1 + E_0$$

iv) $-\epsilon_2 \frac{\partial V_2}{\partial r} \Big|_R + \epsilon_1 \frac{\partial V_1}{\partial r} \Big|_R = 0$

$$-\epsilon_0 (-E_0 P_1 + \sum_l (-l-1) d_l R^{-l-2} \underbrace{P_l}_{\text{basis}}) + \epsilon (\sum_l (l) a_l R^{l-1} \underbrace{P_l}_{\text{basis}}) = 0$$

$$l \neq 1: \boxed{-\epsilon_0 (-l-1) d_l R^{-l-2} + \epsilon (l) a_l R^{l-1} = 0} \Rightarrow a_l = d_l = 0.$$

$$\begin{pmatrix} \times & \times \\ \times & \times \end{pmatrix} \begin{pmatrix} a_l \\ d_l \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$l=1: \epsilon_0 E_0 + 2\epsilon_0 \underline{d_1/R^3} + \epsilon a_1 = 0 \quad d_1/R^3 = a_1 + E_0$$

$$\epsilon_0 E_0 + 2\epsilon_0 (a_1 + E_0) + \epsilon a_1 = 0 \quad -(2\epsilon_0 + \epsilon)$$

$$a_1 = \frac{-3\epsilon_0}{2\epsilon_0 + \epsilon} E_0 \quad \frac{d_1}{R^3} = \left(\frac{-3\epsilon_0}{2\epsilon_0 + \epsilon} + \frac{2\epsilon_0 + \epsilon}{2\epsilon_0 + \epsilon} \right) E_0$$

$$= \left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \right) E_0$$

Final Solution

$$V_1 = \frac{-3\epsilon_0}{2\epsilon_0 + \epsilon} z = \frac{-3}{3 + \chi_e} E_0 r \cos \theta \quad V_2 = \underbrace{-E_0 z}_{\parallel} + \left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \right) R^3 \underbrace{\left[\frac{\cos \theta}{r^2} \right]}_{\parallel} E_0$$

$$= \underbrace{-E_0 z}_{\vec{E}_0} - \underbrace{\frac{\chi_e}{3 + \chi_e} E_0 r \cos \theta}_{\vec{E}_0}$$

$$\vec{E}_2 = \vec{E}_0 + \frac{\vec{P} \cos \theta}{4\pi\epsilon_0 r^2}$$

$$\vec{P} = \frac{4\pi R^3}{3} \cdot 3 \underbrace{\left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \right) \epsilon_0 E_0}_{\frac{d\vec{P}}{dt} = \vec{P}}$$

$$\text{so } \vec{P} = -3\epsilon_0 E_b = \frac{3\chi_e}{3 + \chi_e} \epsilon_0 \vec{E}_0 = \chi_e \epsilon_0 \vec{E}_{\text{tot}}$$

$$\vec{E}_{\text{tot}} = \frac{\vec{P}}{\chi_e} = \frac{3}{3 + \chi_e} \vec{E}_0 \quad [\text{inside}]$$

$$\vec{D} = \epsilon_0 \vec{E}_{\text{tot}} + \vec{P} = \frac{3\epsilon}{3 + \chi_e} \vec{E}_0$$

$$\vec{P} = 3 \left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \right) \epsilon_0 \vec{E}_0$$

$$= 3 \frac{\chi}{3 + \chi} \epsilon_0 \vec{E}_0$$