Magnetic Multipoles

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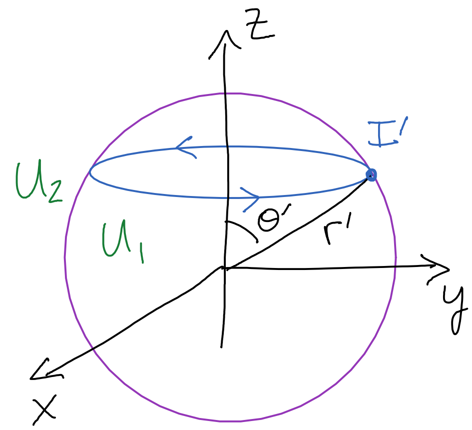
11:08 AM

$*$ Partial Differential Equation : Maxwell Eq's $\Rightarrow$ Laplace Eq.

$$\boxed{\begin{array}{l}\nabla\times\vec{H}=\vec{J}\\ \nabla\cdot\vec{B}=0\\ B=\mu H\end{array}} \to 0 \Rightarrow \quad \vec{H}=-\nabla U$$

$$\Rightarrow -\nabla\cdot(\mu\nabla U)=0$$

or $\boxed{\nabla^2 U=0}$



$$\boxed{\begin{array}{l}U_1=\sum_\ell (A_\ell r^\ell+\cancel{B_\ell r^{-\ell-1}})P_\ell\\ U_2=\sum_\ell(\cancel{C_\ell r^\ell}+D_\ell r^{-\ell-1})P_\ell\end{array}}$$ general solution

$*$ Boundary conditions – always derived the same way.

$\nabla\to\Delta\hat{n}$

$\frac{1}{dt}\to\frac{1}{da}$

$$\int_{-\varepsilon}^{\varepsilon}dn\left(\hat{n}\frac{\partial}{\partial n}+\dots\right)\times\vec{H}=\int_{-\varepsilon}^{\varepsilon}dn\vec{K}\,\delta(n)$$

$$\hat{n}\times\Delta\vec{H}=\vec{K}$$

$\int_a^b dx=\Delta x$

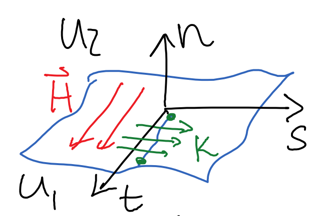
$\hat{n}\cdot\Delta\vec{B}=0$ $\qquad$ $B_{2n}-B_{1n}=0$

$\hat{n}\times\Delta\vec{H}=\vec{K}$ $\qquad$ $H_{2t}-H_{1t}=K_s$

$$\boxed{\begin{array}{l}-\mu_2\frac{\partial U_2}{\partial n}+\mu_1\frac{\partial U_1}{\partial n}=0\\ \int dt\left[-\frac{\partial U_2}{\partial t}+\frac{\partial U_1}{\partial t}=K_s\right]\end{array}}$$ boundary conditions

$-\Delta U=I$

$*$ Application to azimuthal currents on a sphere:



$\hat{n}\times\hat{t}=\hat{s}$

1) $-\frac{\partial U_2}{\partial r}\Big|_{r'}+\frac{\partial U_1}{\partial r}\Big|_{r'}=0$

$$\sum_\ell\left[D_\ell(-\ell-1)r'^{-\ell-2}+A_\ell(\ell)r'^{\ell-1}\right]\underbrace{P_\ell}_{\text{basis}}=0$$

$$D_\ell=-\frac{\ell}{\ell+1}r'^{2\ell+1}A_\ell$$

2) $-\frac{\partial U_2}{r\partial\theta}\Big|_{r'}+\frac{\partial U_1}{r\partial\theta}\Big|_{r'}=K_\phi$

$$\sum\left(-\frac{1}{r'}D_\ell r'^{-\ell-1}+\frac{1}{r'}A_\ell r'^\ell\right)P_\ell'(c_\theta)(-s_\theta)=K_\phi$$

$$\sum_l -A_l r'^{\,l-1}\frac{2l+1}{l+1}\underbrace{P_l'(c_\theta)s_\theta}_{\text{basis fn.}} = K_\phi(\theta)$$

* Orthogonality relation for $P_l'(c_\theta)\cdot s_\theta = P_l'(x)\sqrt{1-x^2}$:

note: $P_l^m = (-1)^m(1-x^2)^{m/2}\left(\frac{d}{dx}\right)^m P_l(x) \;\Rightarrow\; P_l^1(x) = -(1-x^2)^{1/2}P_l'(x)$

$$\int_{-1}^{1} P_{l'}^m(x)P_l^m(x)\,dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!} \;\to\; \frac{2l(l+1)}{2l+1} \quad \text{for } m=1 \; [P_l^1(x)]$$

thus $\langle P_{l'}'|P_l'\rangle \equiv \int_{-1}^{1} P_{l'}'(x)P_l'(x)(1-x^2)\,dx = \langle P_{l'}^1|P_l^1\rangle = \frac{2l(l+1)}{2l+1}\delta_{ll'}$

$$\sum_l -A_l\frac{2l+1}{l+1}r'^{\,l-1}\int_0^\pi P_{l'}'(c_\theta)P_l'(c_\theta)\,s_\theta^3\,d\theta = \int_0^\pi P_{l'}'(c_\theta)K_\phi(\theta)\,s_\theta^2\,d\theta$$

$$= -A_{l'}\cancel{\frac{2l'+1}{l'+1}}\,r'^{\,l-1}\cdot\frac{2l'\cancel{(l'+1)}}{\cancel{2l'+1}}$$

$$\boxed{\begin{aligned} A_l &= \frac{-r'^{\,1-l+1}}{2l}\int_0^\pi P_l'(c_\theta)K_\phi(\theta)\,s_\theta^2\,d\theta \\ D_l &= +\frac{r'^{\,l+2}}{2(l+1)}\int_0^\pi P_l'(c_\theta)K_\phi(\theta)\,s_\theta^2\,d\theta \end{aligned}}$$

* Reduction to a current loop

Current loop: $K_\phi(\theta) = \frac{I'}{r'}\delta(\theta-\theta')$

$$\int_0^\pi P_l'(c_\theta)K_\phi(\theta)\,s_\theta^2\,d\theta = \frac{I's_{\theta'}^2}{r'}P_l'(c_{\theta'})$$

$$\boxed{\begin{aligned} q_l \equiv A_l &= \frac{-I's_{\theta'}^2}{2l\,r'^{\,l}}P_l'(c_{\theta'}) & p_l \equiv 4\pi D_l &= \frac{2\pi I' s_{\theta'}^2 r'^{\,l+1}}{l+1}P_l'(c_{\theta'}) \\ U_{int} &= \sum_l q_l r^l P_l(c_\theta) & U_{ext} &= \sum_l \frac{p_l}{4\pi r^{l+1}}P_l(c_\theta) \end{aligned}}$$

$$\Phi_{in} = \sum_\ell q_\ell r^\ell P_\ell(\cos\theta) \qquad \Phi_{ext} = \sum_\ell \frac{p_\ell}{4\pi r^{\ell+1}} P_\ell(\cos\theta)$$

[external multipole] [internal multipole]

Note: $q_\ell^E = \frac{p_\ell^E}{4\pi r'^{2\ell+1}}$ $\qquad q_\ell^M = \frac{-(\ell+1)\, p_\ell^M}{\ell\, 4\pi r'^{2\ell+1}}$

[electric] [magnetic]

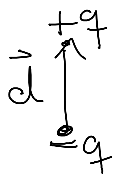
* external $[q_\ell]$ vs. internal $[p_\ell]$ moments.

* Magnetic dipole moment:

$$m = p_1 = \frac{2\pi}{2} I' s^2 = I' A'$$

$P_1 = x \Rightarrow P_1' = 1$

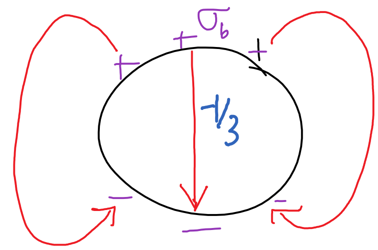
Electric vs. Magnetic



$\vec{p} = q\vec{d}$

$\vec{A} = \pi s^2 \hat{z}$

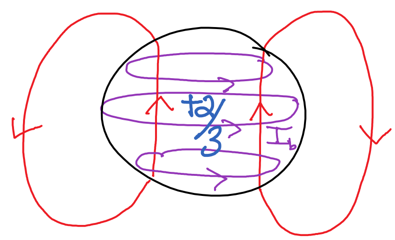
$\vec{m} = I\vec{A}$



* Dipole fields:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3\vec{p}\cdot\hat{r}\,\hat{r} - \vec{p}\right] - \frac{1}{3}\vec{p}\,\delta^3(\vec{r})$$

$$\vec{B} = \frac{\mu_0}{4\pi r^3}\left[3\vec{m}\cdot\hat{r}\,\hat{r} - \vec{m}\right] + \frac{2}{3}\vec{m}\,\delta^3(\vec{r})$$



* Legendre polynomials & derivatives:

$P_0 = 1 \quad P_2 = \frac{1}{2}(3c^2-1) \quad P_4 = \frac{1}{8}(35c^4 - 30c^2 + 3)$

$P_1 = c \quad P_3 = \frac{1}{2}(5c^3 - 3c) \quad P_5 = \frac{1}{8}(63c^5 - 70c^3 + 15c)$

$P_0' = 0 \quad P_2' = 3c \quad P_4' = \frac{5}{2}(7c^3 - 3c)$

$P_1' = 1 \quad P_3' = \frac{3}{2}(5c^2 - 1) \quad P_5' = \frac{5}{8}(63c^4 - 52c^2 + 3)$

* Helmholtz coil:

2 symmetric coils @ $\theta' = \theta_1$

$\Rightarrow$ even multipoles $P_l'(c_\theta)$ vanish: $l = 2, 4, 6, \ldots$

$l=1$: dipole moment

- we have one degree of freedom to cancel out a higher moment: $l=3$

$$q_3 = \frac{I_1 s_1^2}{2 \cdot 3 r'^3} P_3'(c_1) = 0 \quad \text{if } P_3' = \frac{3}{2}(5c_1^2 - 1) = 0 \quad \boxed{c_1 = \frac{1}{2}}$$

* Maxwell coil:

add a centre ring with current I_0 $c_0 = 0$ r'
to the symmetric pair with I_1; c_1 also r'

- use I_0 to fix the magnitude, then we have two parameters I_1, c_1 to null q_3 and q_5

$$q_3 = \frac{-1}{6r'^3}\left[I_0 \cancel{s_0^2}^{1} P_3'(c_0) + 2 I_1 s_1^2 P_3'(c_1)\right] = 0$$

$$-\frac{6}{6r'^3} \cdot \frac{3}{2}\left[I_0(5\cancel{c_0^2}^{0} - 1) + 2 I_1 s_1^2 (5c_1^2 - 1)\right] = 0$$

$$-I_0 + 2I_1(1 - c_1^2)(5c_1^2 - 1) = 0 \qquad [1]$$

$$q_5 = \frac{-1}{10r'^5}\left[I_0 \cancel{s_0^2}^{1} P_5'(c_0) + 2 I_1 s_1^2 P_5'(c_1)\right] = 0$$

$$\frac{-1}{10r'^5} \frac{5}{8}\left[I_0(63\cancel{c_0^4}^{0} - 52\cancel{c_0^2}^{0} + 3) + 2I_1 s_1^2 (63c_1^4 - 52c_1^2 + 3)\right] = 0$$

$$3I_0 + 2I_1(1 - c_1^2)(63c_1^4 - 52c_1^2 + 3) = 0 \qquad [2]$$

$3\cdot[1] + [2]$: $$0 I_0 + 2I_1(1 - c_1^2)\left[3(5c_1^2 - 1) + (63c_1^4 - 52c_1^2 + 3)\right] = 0$$

$$3(5c_1^2 + 21c_1^4 - 14c_1^2) = 9c_1^2(7c_1^2 - 3) = 0$$

$c_1^2 = 3/7 \quad s_1^2 = 4/7 \qquad I_0/I_1 = 2s_1^2(5c_1^2 - 1) = 8^2/7^2$

check [2]: $3 \cdot \frac{8^2}{7^2} I_1 + 2I_1 (4/7)(63\,\frac{3^2}{7^2} - 52\,\frac{3}{7} + 3)$

$$= \frac{3 \cdot 8 I_1}{7^2}\left[8 + (3 \cdot 3^2 - 14 \cdot 3 + 7)\right] = 0 \quad \checkmark$$

Solution: put a coil pair at angle $\cos\theta = \sqrt{3/7}$ and one in the center with $64/47 \times$ more current, all on the surface of a sphere.

* Gradient coil:

2 coils with opposite current
By symmetry odd multipoles vanish.
q_2 is the gradient, adjust $c_1 = \cos(\theta_1)$ to null q_4

$$q_4 = \frac{2 I_1 s_1^2}{8 r'^4} P_4'(c_1) = 0 \qquad 7c_1^2 - 3 = 0 \qquad c_1^2 = 3/7$$

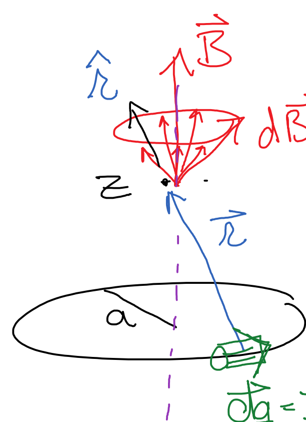
This is the same configuration as the Maxwell coil without the center coil.

* External multipole:

$$\vec{H} = \int \frac{d\vec{q} \times \vec{r}}{4\pi r^3} = \int_{\phi=0}^{2\pi} \frac{I(\cancel{z}\hat{s} + a\hat{z})\, a\, d\phi}{4\pi(z^2 + a^2)^{3/2}}$$

$$= \frac{2\pi I a^2 \hat{z}}{4\pi(z^2+a^2)^{3/2}} = \frac{\vec{m}}{2\pi r^3}$$

where $\vec{m} = I\vec{A} = I \pi a^2 \hat{z}$



$$d\vec{l} \times \vec{r} = a\,d\phi\,\hat{\phi} \times (z\hat{z} - a\hat{s})$$
$$= (z\hat{s} + a\hat{z})\, a\, d\phi$$

$$U_{int} = \sum_\ell q_\ell r^\ell P_\ell(c_\theta) \qquad q_\ell = \frac{-I' s_{\theta'}^2}{2\ell r'^\ell} P_\ell'(c_{\theta'})$$

$$U_{int}^1 = q_1 z = \frac{m z}{2\pi r'^3} \qquad q_1 = \frac{-I'\pi(r' s_{\theta'})^2}{2\pi r'^3} = \frac{-I'A}{2\pi r'^3} = \frac{-m}{2\pi r'^3} = \frac{-p_1}{2\pi r'^3}$$

$$\vec{H} = -\nabla U = \frac{\vec{m}}{2\pi r'^3} \checkmark$$

not 4π? because of $\frac{-(\ell+1)}{\ell} = -2$