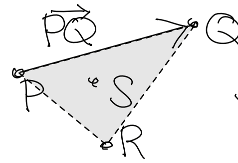
Section 1.4 - Affine Spaces

* Affine Space - linear space of points

	POINTS	vs	VECTORS
~ operations	$Q - P = \vec{V}$ $P + \vec{V} = Q$		$\vec{W} = \alpha\vec{u} + \beta\vec{v}$



$S = \alpha P + \beta Q + \gamma R$

$\alpha + \beta + \gamma = 1$

~ points are invariant under translation of the origin, but coordinates depend on the origin

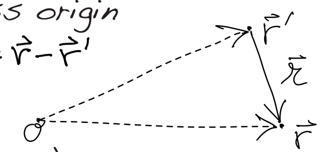
~ a point may be specified by its 'position vector' (arrow from the origin to the point)

cumbersome picture: many meaningless arrows from a meaningless origin

position vector: field point $\vec{r} = (x, y, z)$ source pt $\vec{r}' = (x', y', z')$

displacement vector: $\vec{\mathscr{r}} \equiv \vec{r} - \vec{r}'$

differential: $d\vec{l} = \frac{\partial\vec{r}}{\partial q} dq = \vec{b}\, dq$



~ the only operation on points is a weighted average (affine combination)

weight $w=0$ forms a vector and $w=1$ forms a point

		vs	
~ transformation:	affine	vs	linear
~ basis (independent):	$N+1$	vs	N
~ decomposition:	coordinates	vs	components

- they appear the same for cartesian systems!
- coordinates are scalar fields $q^i(\vec{r})$
- they parametrize space

$$\begin{pmatrix} R & \vec{t} \\ 000 & 1 \end{pmatrix}\begin{pmatrix} \vec{r} \\ 1 \end{pmatrix} = \begin{pmatrix} R\vec{r} + \vec{t} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} R & \vec{t} \\ 000 & 1 \end{pmatrix}\begin{pmatrix} \vec{v} \\ 0 \end{pmatrix} = \begin{pmatrix} R\vec{v} \\ 0 \end{pmatrix}$$

* Rectangular, Cylindrical and Spherical coordinate transformations

~ math: 2-d -> N-d physics: 3d + azimuthal symmetry

~ singularities on z-axis and origin

$S_\theta \equiv \sin\theta$

$C_\theta \equiv \cos\theta$

$$(\hat{s}, \hat{\phi}, \hat{z}) = (\hat{x}, \hat{y}, \hat{z}) \underbrace{\begin{pmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{R_{\hat{z}}(\phi)}$$

$x = s \cdot C_\phi$

$y = s \cdot S_\phi$

$s = r \cdot S_\theta$

$z = r \cdot C_\theta$

rect.		cyl.		sph.
x	$=$	$s \cdot C_\phi$	$=$	$r \cdot S_\theta \cdot C_\phi$
y	$=$	$s \cdot S_\phi$	$=$	$r \cdot S_\theta \cdot S_\phi$
z	$=$	z	$=$	$r \cdot C_\theta$

$$(\hat{r}\,\hat{\theta}\,\hat{\phi}) = (\hat{s}\,\hat{\phi}\,\hat{z}) \underbrace{\begin{pmatrix} S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \\ C_\theta & -S_\theta & 0 \end{pmatrix}}_{R_{\hat{\phi}}(\theta)} = (\hat{x}\,\hat{y}\,\hat{z}) \underbrace{R_{\hat{z}}(\phi) \cdot R_{\hat{\phi}}(\theta)}_{R(\theta,\phi)}$$

$$R_{\hat{z}}(\phi) \cdot R_{\hat{\phi}}(\theta) = \begin{pmatrix} S_\theta C_\phi & C_\theta C_\phi & -S_\phi \\ S_\theta S_\phi & C_\theta S_\phi & C_\phi \\ C_\theta & -S_\theta & 0 \end{pmatrix}$$

$d\vec{l}_{rec} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$

$d\vec{l}_{cyl} = \hat{s}\,ds + \hat{\phi}\,s\,d\phi + \hat{z}\,dz$

$d\vec{l}_{sph} = \hat{r}\underbrace{dr}_{dl_r} + \hat{\theta}\underbrace{r\,d\theta}_{dl_\theta} + \hat{\phi}\underbrace{r\sin\theta\,d\phi}_{dl_\phi}$

$d\vec{a}_{rec} = \hat{x}\,dy\,dz + \hat{y}\,dz\,dx + \hat{z}\,dx\,dy$

$d\vec{a}_{cyl} = \hat{s}\,s\,d\phi\,dz + \hat{\phi}\,dz\,ds + \hat{z}\,ds\,s\,d\phi$

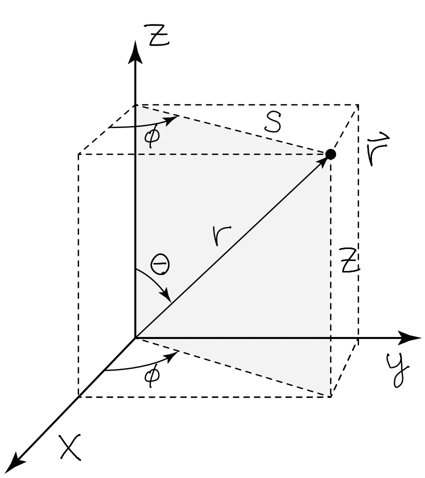
$d\vec{a}_{sph} = \hat{r}\,r\,d\theta\,r\sin\theta\,d\phi + \hat{\theta}\,r\sin\theta\,d\phi\,dr + \hat{\phi}\,dr\,r\,d\theta$

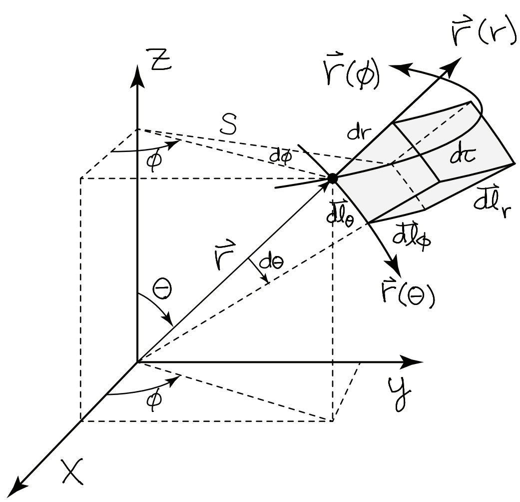
$d\tau_{rec} = dx\,dy\,dz$

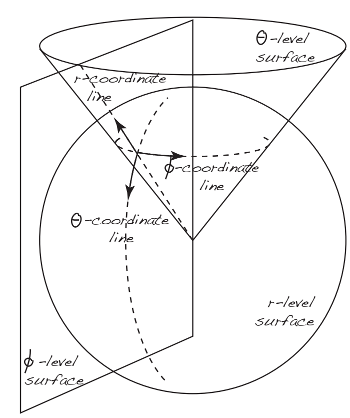
$d\tau_{cyl} = ds \cdot s\,d\phi \cdot dz$

$d\tau_{sph} = dr \cdot r\,d\theta \cdot r\sin\theta\,d\phi$

$= r^2\,dr\,d\Omega$







Curvilinear Coordinates

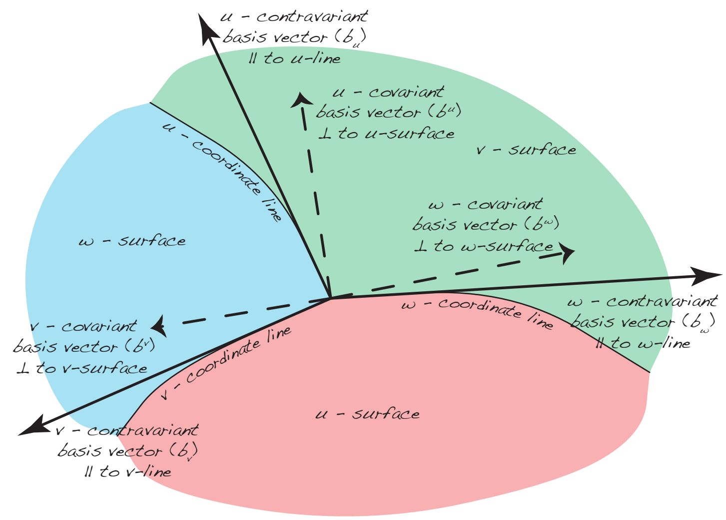
* coordinate surfaces and lines

~ each coordinate is a scalar field $q(\vec r)$

~ coordinate surfaces: constant q^i

~ coordinate lines: constant q^j, q^k

* coordinate basis vectors



$q^i \sim \{u, v, w\}$ ~ generalized coordinates

$\vec b_i = \left(\frac{\partial \vec r}{\partial q^i}\right)_{q^j, q^k} \sim \{\hat u f, \hat v g, \hat w h\}$ ~ contravariant basis

$\vec b^i = \nabla q^i \sim \{\hat u/f, \hat v/g, \hat w/h\}$ ~ covariant basis

$h_i = |\vec b_i| \sim \{f, g, h\}$ ~ scale factor

$\hat e_i = \vec b_i / h_i \sim \{\hat u, \hat v, \hat w\}$ ~ unit vector

$g_{ij} = \vec b_i \cdot \vec b_j \sim \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix}$ ~ metric (dot product)

$\vec\Gamma_{ij} = \frac{\partial \vec b_j}{\partial q^i} = \vec b_k \Gamma^k_{ij}$ ~ Christoffel symbols - derivative of basis vectors

* differential elements (orthogonal coords)

$$d\vec l = \frac{\partial \vec r}{\partial q^1} dq^1 + \frac{\partial \vec r}{\partial q^2} dq^2 + \frac{\partial \vec r}{\partial q^3} dq^3 = \vec b_i\, dq^i$$

$$= \hat e_1 \underbrace{h_1 dq^1}_{dl_1} + \hat e_2 \underbrace{h_2 dq^2}_{dl_2} + \hat e_3 \underbrace{h_3 dq^3}_{dl_3}$$

$$d\vec a = \tfrac{1}{2} d\vec l \times d\vec l = \begin{vmatrix} \hat e_1 & \hat e_2 & \hat e_3 \\ h_1 dq^1 & h_2 dq^2 & h_3 dq^3 \\ h_1 dq^1 & h_2 dq^2 & h_3 dq^3 \end{vmatrix}$$

$$= \hat e_1 \cdot h_2 dq^2 \cdot h_3 dq^3 + \hat e_2 \cdot h_3 dq^3 \cdot h_1 dq^1 + \hat e_3 \cdot h_1 dq^1 \cdot h_2 dq^2$$

$$d\tau = \tfrac{1}{3} d\vec l \cdot d\vec a = \tfrac{1}{2} d\vec l \cdot d\vec l \times d\vec l = h_1 dq^1 \cdot h_2 dq^2 \cdot h_3 dq^3$$

* example $\quad (c_\phi \equiv \cos\phi)$

$$x = s\, c_\phi \qquad dx = c_\phi\, ds - s\, s_\phi\, d\phi$$

$$y = s\, s_\phi \qquad dy = s_\phi\, ds + s\, c_\phi\, d\phi$$

$$d\vec l = \hat x\, dx + \hat y\, dy = (\hat x c_\phi + \hat y s_\phi) ds + (\hat x s_\phi - \hat y c_\phi) s\, d\phi$$

$$= \hat s\, ds + \hat\phi\, s\, d\phi \qquad (\hat s \;\; \hat\phi) = (\hat x \;\; \hat y)\begin{pmatrix} c_\phi & -s_\phi \\ s_\phi & c_\phi \end{pmatrix}$$

$$s^2 = x^2 + y^2 \qquad 2s\,ds = 2x\,dx + 2y\,dy$$

$$y = x \tan\phi \qquad dy = dx \tan\phi + x \sec^2\phi\, d\phi$$

$$d\phi = \frac{-y}{s^2} dx + \frac{x}{s^2} dy$$

$$\nabla s = \frac{x}{s}\hat x + \frac{y}{s}\hat y = c_\phi \hat x + s_\phi \hat y = \hat s$$

$$\nabla\phi = \frac{-y}{s^2}\hat x + \frac{x}{s^2}\hat y = \frac{-s_\phi \hat x + c_\phi \hat y}{s} = \frac{\hat\phi}{s}$$

* formulas for vector derivatives in orthogonal curvilinear coordinates

$$df = \frac{\partial f}{\partial q^i} dq^i = \frac{\partial f}{h_i \partial q^i} \cdot h_i dq^i = \nabla f \cdot d\vec l \qquad \nabla f = \frac{df}{d\vec r} = \frac{\hat e_i}{h_i}\frac{\partial}{\partial q^i} f$$

$$d(\vec A \cdot d\vec l) = d(A_k h_k dq^k) = \frac{\partial}{\partial q^j}(h_k A_k)\, dq^j dq^k$$

$$= \varepsilon_{ijk} \frac{\partial (h_k A_k)}{h_j h_k \partial q^k} da_i = (\nabla \times \vec A) \cdot d\vec a$$

$$\nabla \times \vec A = \frac{d(d\vec r \cdot \vec A)}{d^2 \vec r} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat e_1 & h_2 \hat e_2 & h_3 \hat e_3 \\ \partial/\partial q^1 & \partial/\partial q^2 & \partial/\partial q^3 \\ h_1 A^1 & h_2 A^2 & h_3 A^3 \end{vmatrix}$$

$$d(\vec B \cdot d\vec a) = d(B_i h_j dq^j h_k dq^k) = \frac{\partial}{\partial q^i}(h_j h_k B_i)\, dq^i dq^j dq^k$$

$$= \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial q} \frac{\partial (h_j h_k B_i)}{\partial q^i} d\tau = \nabla \cdot \vec B\, d\tau$$

$$\nabla \cdot \vec B = \frac{d(d^2\vec r \cdot \vec B)}{d^3 \vec r} = \frac{1}{h_1 h_2 h_3} \sum_i \frac{\partial}{\partial q^i}(h_j h_k B_i)$$

i, j, k cyclic

this formula does not work for $\nabla^2 \vec B \longrightarrow$ $$\nabla^2 f = \frac{1}{h_1 h_2 h_3} \sum_i \frac{\partial}{\partial q_i} \frac{h_j h_k}{h_i} \frac{\partial}{\partial q_i} f$$

instead, use: $\nabla^2 = \nabla\nabla\cdot \; - \nabla\times\nabla\times$