

Section 3.3.2 - Separation of Variables (Spherical)

* same technique as in rectangular coordinates

~ the differential equations are more complex, but we only solve them once

~ boundary conditions are of two types

a) radial - external boundary condition - treated in the same way as cartesian

b) angular - internal to the problem - almost always have the same solution

* key principles:

~ separation of variables

$$V(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi)$$

~ orthogonality of

$$\Theta(\theta) = P_l(\cos \theta)$$

~ boundary conditions

$$r \rightarrow 0, r = a, r \rightarrow \infty$$

* separation of variables - slight twist: solve one eigenvalue at a time $-m^2 V$

$$\nabla^2 V(r, \theta, \phi) = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} V + \underbrace{\frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} V + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} V}_{-l(l+1)V} = -\rho/\epsilon_0 = 0$$

RADIAL EQUATION

$$\frac{d}{dr} r^2 \frac{d}{dr} R(r) = l(l+1) R(r)$$

$$\text{let } R(r) = r^\alpha \quad \alpha(\alpha+1) = l(l+1)$$

$$\alpha = l, -(l+1)$$

$$R(r) = A r^l + B r^{-l-1}$$

POLAR EQUATION ($m=0$)

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \sin \theta \frac{d}{d\theta} \Theta(\theta) = -l(l+1) \Theta(\theta)$$

$$\text{let } x = \cos(\theta) \quad dx = -\sin \theta d\theta$$

$$\Theta(\theta) = P_l(x) \quad \sin \theta d\theta d\phi \rightarrow -dx d\phi$$

$$\frac{d}{dx} (1-x^2) \frac{d}{dx} P_l(x) + l(l+1) P_l(x) = 0$$

$$\Theta(\theta) = P_l(x) = P_l(\cos \theta); \quad Q_l(\cos \theta) \text{ diverges}$$

AZIMUTHAL EQ.

$$\frac{d^2}{d\phi^2} \Phi = -m^2 \Phi$$

$$\Phi(\phi) = e^{im\phi}$$

if $m=0$ then

$$\Phi(\phi) = \text{const}$$

* general solution

$$\nabla^2 V = 0$$

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

* boundary conditions

$$\text{i) at } r=0, \quad \frac{1}{r^{l+1}} \rightarrow \infty \quad \text{so } B_l = 0$$

$$\text{ii) at } r=\infty, \quad r^l \rightarrow \infty \quad \text{so } A_l = 0$$

$$\text{iii) at } r=a, \quad (1) \quad V_0(\theta) = V(a, \theta) = \sum_{l=0}^{\infty} \left(A_l a^l + \frac{B_l}{a^{l+1}} \right) P_l(\cos \theta)$$

$$E_{\text{ext}} = E_0 \hat{x} = -\nabla(-r' \cos \theta)$$

$$(2) \quad \frac{\partial V_0}{\partial r}(\theta) = \frac{\partial V}{\partial r}(a, \theta) = \sum_{l=0}^{\infty} \left(l A_l a^{l-1} - (l+1) \frac{B_l}{a^{l+2}} \right) P_l(\cos \theta)$$

surface boundary at the interface between two regions with surface charge σ

$$\nabla \cdot \epsilon_0 \vec{E} = \rho \Rightarrow \hat{n} \cdot (\vec{E}_2 - \vec{E}_1) = \sigma/\epsilon_0$$

$$\nabla \times \vec{E} = 0 \Rightarrow \hat{n} \times (\vec{E}_2 - \vec{E}_1) = 0$$

$$E_{2n} - E_{1n} = \sigma/\epsilon_0$$

$$E_{2t} - E_{1t} = 0$$

$$\Rightarrow V'_1(a) - V'_2(a) = \sigma/\epsilon_0$$

$$V_1(a) = V_2(a)$$

* properties of the Legendre polynomials

~ Rodrigues formula

~ orthogonality

$$P_l(x) = \frac{1}{2^l l!} \left(\frac{d}{dx} \right)^l (x^2-1)^l \quad l=0, 1, 2, \dots$$

$$\langle P_l | P_{l'} \rangle \equiv \int_{-1}^1 P_l(x) P_{l'}(x) dx = \int_0^\pi P_l(\cos \theta) P_{l'}(\cos \theta) \sin \theta d\theta = \begin{cases} 0 & \text{if } l' \neq l \\ \frac{2}{2l+1} & \text{if } l' = l \end{cases}$$

~ this is only one independent solution

the other solutions $Q(x)$ blows up at the N&S poles ($\theta=0, 2\pi$)

and doesn't satisfy continuity boundary conditions

Problem 3.9

* spherical shell of charge $\sigma = \sigma_0 \sin^2 \theta$

inside region: $V_1(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$

outside region: $V_2(r, \theta) = \sum_{l=0}^{\infty} \left(C_l r^l + \frac{D_l}{r^{l+1}} \right) P_l(\cos \theta)$

boundary conditions:

i) $V_1(0, \theta)$ finite $\Rightarrow B_l = 0$

ii) $V_2(\infty, \theta)$ finite $\Rightarrow C_l = 0$ (let $C_0 = 0$ also)

iii) $V_1(R, \theta) = V_2(R, \theta)$ $\sum_{l=0}^{\infty} (A_l R^l + 0) P_l(\cos \theta) = \sum_{l=0}^{\infty} (0 + \frac{D_l}{R^{l+1}}) P_l$
 $\sum_{l=0}^{\infty} (A_l R^l - \frac{D_l}{R^{l+1}}) P_l(\cos \theta) = 0 \Rightarrow D_l = A_l R^{2l+1}$

iv) $E_{2n} - E_{1n} = \sigma / \epsilon_0$

$$-\frac{\partial V_2}{\partial r} \Big|_R + \frac{\partial V_1}{\partial r} \Big|_R = \frac{\sigma}{\epsilon_0} = \frac{\sigma_0}{\epsilon_0} \sin^2 \theta$$

$$\sum_{l=0}^{\infty} \left(D_l \frac{(l+1)}{R^{l+2}} + A_l \cdot l R^{l-1} \right) P_l(\cos \theta) = \frac{\sigma_0}{\epsilon_0} \sin^2 \theta$$

* $\sum_{l=0}^{\infty} A_l (2l+1) R^{l-1} \cdot P_l(\cos \theta) = \frac{\sigma_0}{\epsilon_0} \sin^2 \theta$

$$(A_0 R^{-1}) P_0 + (A_1 3 R^0) P_1 + (A_2 5 R) P_2 + \dots = \left(\frac{\sigma_0}{\epsilon_0} \frac{2}{3} \right) P_0 + 0 + \left(\frac{\sigma_0}{\epsilon_0} \frac{-2}{3} \right) P_2 + \dots$$

$$A_0 = \frac{2\sigma_0}{3\epsilon_0 R} \quad A_1 = 0 \quad A_2 = \frac{-2\sigma_0 R}{15\epsilon_0}$$

solutions: inside $V_1 = \frac{2\sigma_0}{3\epsilon_0} \left(\frac{1}{R} - \frac{r^2}{5R^3} \frac{1}{2} (3\cos^2 \theta - 1) \right)$

$V_1 = V_2$ @ $r = R$

outside $V_2 = \frac{2\sigma_0}{3\epsilon_0} \left(\frac{1}{R} - \frac{R^2}{5r^3} \frac{1}{2} (3\cos^2 \theta - 1) \right)$

$-V_2' + V_1' = \sigma / \epsilon_0$ @ $r = R$

alternate solution of B.C. iv (use integrals to extract components like in Section 3.2.1)

* $\int_0^\pi P_0(\cos \theta) \cdot \sin^2 \theta \sin \theta d\theta = \int_0^\pi \sin^3 \theta d\theta = \frac{4}{3}$

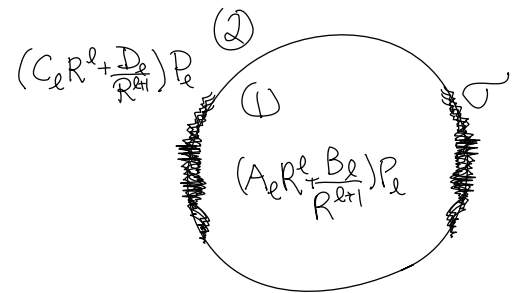
$$\int_0^\pi P_0(\cos \theta) \cdot P_0(\cos \theta) \sin \theta d\theta = \int_0^\pi \sin \theta d\theta = \frac{2}{1}$$

$$\int_0^\pi P_1(\cos \theta) \cdot \sin^2 \theta \sin \theta d\theta = \int_0^\pi \cos \theta \cdot \sin^3 \theta d\theta = 0$$

$$\int_0^\pi P_1(\cos \theta) \cdot P_1(\cos \theta) \sin \theta d\theta = \int_0^\pi \cos^2 \theta \cdot \sin \theta d\theta = \frac{2}{3}$$

$$\int_0^\pi P_2(\cos \theta) \cdot \sin^2 \theta \sin \theta d\theta = \int_0^\pi \frac{1}{2} (3\cos^2 \theta - 1) \cdot \sin^3 \theta d\theta = \frac{-4}{15}$$

$$\int_0^\pi P_2(\cos \theta) \cdot P_2(\cos \theta) \sin \theta d\theta = \int_0^\pi \frac{1}{4} (3\cos^2 \theta - 1)^2 \cdot \sin \theta d\theta = \frac{2}{5}$$



4x∞ unknowns
4 B.C.'s.

$$\begin{aligned} \sin^2 \theta &= 1 - \cos^2 \theta \\ &= -\cos^2 \theta + \frac{1}{2} + \frac{2}{3} \\ &= -\frac{2}{3} P_2(\cos \theta) + \frac{2}{3} P_0(\cos \theta) \end{aligned}$$