Section 3.4 - Multipoles

* *binomial expansion*

$$(a+b)^0 = 1$$
$$(a+b)^1 = a + b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = \binom{4}{0}a^4b^0 + \binom{4}{1}a^3b^1 + \binom{4}{2}a^2b^2 + \binom{4}{3}a^1b^3 + \binom{4}{4}a^0b^4$$

* *Pascal's triangle*

$n=0$	1
$n=1$	1 1
$n=2$	1 2 1
$n=3$	1 3 3 1
$n=4$	1 4 6 4 1
	$\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

~ *general form*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdots k}$$

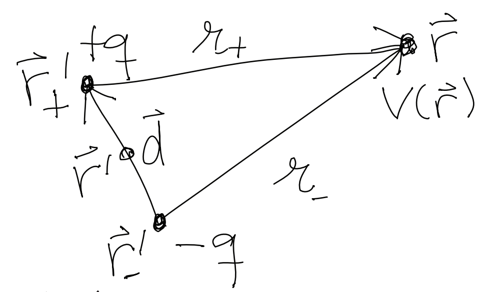
~ *if* $n \to \alpha$ *(any real number), then the series does not terminate unless* $\alpha = 0,1,2,\ldots$

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{1\cdot 2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{1\cdot 2\cdot 3}x^3 + \cdots$$

~ *example:*

$$\frac{1}{1-x} = 1-(-x) + \frac{-1\cdot -2}{1\cdot 2}(-x)^2 + \frac{-1\cdot -2\cdot -3}{1\cdot 2\cdot 3}(-x)^3 + \ldots$$
$$= 1 + x + x^2 + x^3 + \ldots \qquad \textit{for radius of convergence } |x|<1$$

* *2-pole expansion*



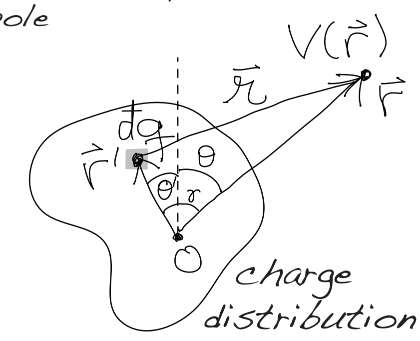
$$V(\vec r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right)$$

$$\vec r_\pm = \vec r \mp \tfrac{1}{2}\vec d \qquad r_\pm^2 = r^2 \mp r d\cos\theta + \tfrac{1}{4}d^2$$

$$\frac{1}{r_\pm} \cong \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-\frac{1}{2}} \qquad \approx \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta + \ldots\right)$$

$$V(\vec r) = \frac{qd\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{\vec p\cdot\vec r}{4\pi\varepsilon_0 r^3} \qquad \boxed{\vec p = q\vec d} \quad \textit{electric dipole moment}$$

* *general axial-symmetric multipole expansion*



$$r^2 = (\vec r - \vec r')^2 = r^2\left(1 - 2\frac{r'}{r}\cos\gamma + \left(\frac{r'}{r}\right)^2\right) \equiv r^2(1+\varepsilon)$$

$$\frac{1}{r} = \frac{1}{r}(1+\varepsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \ldots\right)$$

$$= \frac{1}{r}\left(1 + \frac{r'}{r}\cos\gamma + \frac{r'^2}{r^2}\frac{1}{2}(3\cos^2\gamma - 1) + \frac{r'^3}{r^3}\frac{1}{2}(5\cos^3\gamma - 3\cos\gamma) + \ldots\right.$$

$$= \frac{1}{r}\left(P_0(\cos\gamma) + \frac{r'}{r}P_1(\cos\gamma) + \frac{r'^2}{r^2}P_2(\cos\gamma) + \cdots \right. \Rightarrow \boxed{\sum_{l=0}^{\infty}\frac{r'^l}{r^{l+1}}P_l(\cos\theta)\,P_l(\cos\theta)}$$

(addition formula for azimuthally symmetry)

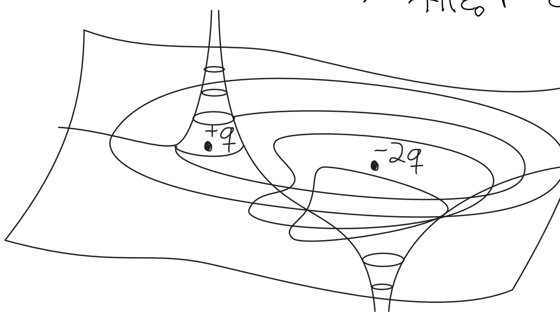
$$V(\vec r) = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}P_l(\cos\theta)\underbrace{\int dq\, r'^l P_l(\cos\theta')}_{Q^{(l)}_{int}}$$

multipole potential — $Q^{(l)}_{int}$ *electric multipole (monopole, dipole, quadrupole)*

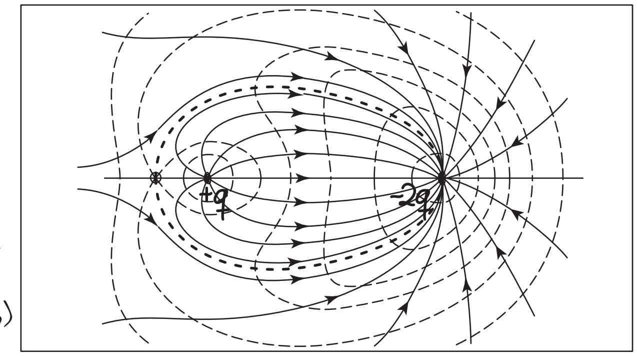
~ $Q^{(l)}_{int}$ *are coefficients of the general solution of Laplace equation in spherical coords*

* *monopole* $V(\vec{r}) \approx \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int dq' = \frac{q}{4\pi\varepsilon_0 r}$

$q \equiv \int dq'$

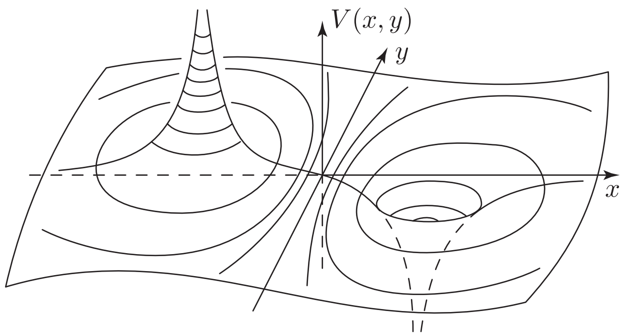


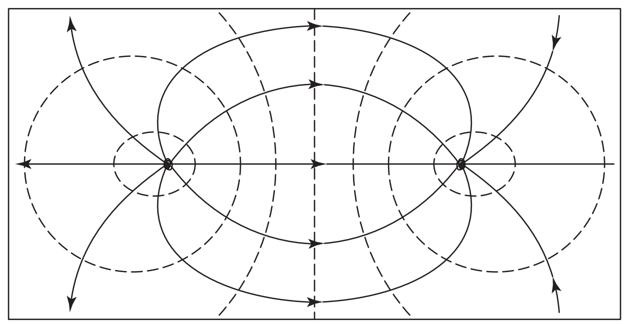
effective monopole (dominated by -2q far from the origin)



* *dipole* $V_1(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^2}\int dq'\, r'\cos\gamma = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3}$ $\quad \vec{p} = \int dq'\,\vec{r}' \quad$ $\vec{r}\cdot\vec{r}' = r\,r'\cos\gamma$

if $q = \int dq' = 0$ *then* $T_{\vec{a}}[\vec{p}] = \int dq'(\vec{r}'-\vec{a}) = \int dq'\,\vec{r}' - \vec{a}\overbrace{\int dq'}^{0} = \vec{p}$





* *quadrupole*

$$V_2(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^3}\int dq'\, r'^2\frac{1}{2}(3\cos^2\gamma - 1) = \frac{1}{4\pi\varepsilon_0 r^5}\int dq'\,\frac{1}{2}(3(\vec{r}\cdot\vec{r}')^2 - r^2)$$

$Q_{xx}+Q_{yy}+Q_{zz}=0$

$$= \frac{\frac{1}{2}\vec{r}\cdot\overleftrightarrow{Q}\cdot\vec{r}}{4\pi\varepsilon_0 r^5}$$

$$\overleftrightarrow{Q} = \int dq'(3\vec{r}'\vec{r}' - I r'^2) = \int dq'\begin{pmatrix} 3x'^2-r'^2 & 3x'y' & 3x'z' \\ 3y'x' & 3y'^2-r'^2 & 3y'z' \\ 3z'x' & 3z'y' & 3z'^2-r'^2 \end{pmatrix}$$

$$Q_{ij} = \int dq'(3r_i'r_j' - \delta_{ij}r'^2)$$

$$T_{\vec{a}}[\overleftrightarrow{Q}] = \int dq'\, 3(\vec{r}'-\vec{a})(\vec{r}'-\vec{a}) - (\vec{r}'-\vec{a})^2 I$$

$$= \int dq'(3\vec{r}'\vec{r}' - r'^2 I) - 3(\vec{r}'\vec{a} + \vec{a}\vec{r}' - \vec{a}\vec{a}) + (2\vec{r}\cdot\vec{a} + \vec{a}^2)I$$

$$= \overleftrightarrow{Q}_0 - [3(\vec{p}\vec{a} + \vec{a}\vec{p}) - 2\vec{p}\cdot\vec{a}I] + [3\vec{a}\vec{a} - \vec{a}^2 I]q$$

$\overleftrightarrow{Q} = 3\sum_i \vec{p}_i\vec{a}_i + \vec{a}_i\vec{p}_i - 2\vec{p}_i\cdot\vec{a}_i I$ *dipoles* $\vec{p}_i$ *at positions* $\vec{a}_i$