Section 5.2 - Biot-Savart Law

* review:

- ~ charge element (scalar): $dq \sim \lambda dl \sim \sigma da \sim \rho d\tau$ $\Big\} \times \vec{v}$
- ~ current element (vector!): surface, volume current density $dq\,\vec{v} \sim I d\vec{l} \sim \vec{K} da \sim \vec{J} d\tau$

$$I = \Phi_J = \int \vec{J} \cdot d\vec{a}$$

- ~ steady currents: analog of electrostatic stationary charges $\Delta I = \frac{dQ}{dt} = 0 \qquad \nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} = 0$

* electrostatic vs. magnetostatic force laws

- ~ definition of "force" fields E, B (vs. "source" fields D, H, see next chapter)
- ~ fields mediate force from one charge (current) to another (action at a distance)
- ~ experiment by Oersted defined direction of field, Ampere defined magnitude

~ Coulomb Law (electric)

$$\vec{F}_e = \frac{1}{4\pi\varepsilon_0} \int_{\vec{r}} \int_{\vec{r}'} dq\, \underbrace{dq' \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}} \equiv \int dq\, \vec{E} \qquad \vec{E} \equiv \frac{1}{4\pi\varepsilon_0} \int_{\vec{r}'} \frac{dq'\,\hat{\mathscr{r}}}{\mathscr{r}^2} = -\nabla \underbrace{\frac{1}{4\pi\varepsilon_0} \int_{\vec{r}'} \frac{dq}{\mathscr{r}}} = -\nabla V$$

~ Biot-Savart Law (magnetic)

$$\vec{F}_m = \frac{-\mu_0}{4\pi} \oint_{\vec{r}} \oint_{\vec{r}'} I d\vec{l} \cdot \underbrace{I' d\vec{l}' \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}} = \oint I' d\vec{l} \times \vec{B} \qquad \vec{B} \equiv \frac{\mu_0}{4\pi} \oint_{\vec{r}'} \frac{I' d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2} = ?$$

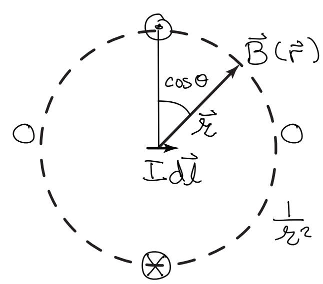
~ proof: $\quad A \times (B \times C) = B\,(A \cdot C) - C\,(A \cdot B)$

$$\vec{F}_m = \oint I d\vec{l} \times \oint' \frac{\mu_0}{4\pi} I' d\vec{l}' \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0}{4\pi} \oint \oint' I' d\vec{l}' \underbrace{\left(I d\vec{l} \cdot -\nabla \frac{1}{\mathscr{r}}\right)}_{\oint I\, d\frac{1}{\mathscr{r}} = 0} - \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \left(I d\vec{l} \cdot I' d\vec{l}'\right)$$

~ combined: Lorentz force law

$$\vec{F} = \int dq\,(\vec{E} + \vec{v} \times \vec{B}) = \int d\tau\,(\rho \vec{E} + \vec{J} \times \vec{B})$$

directional inverse-square force law

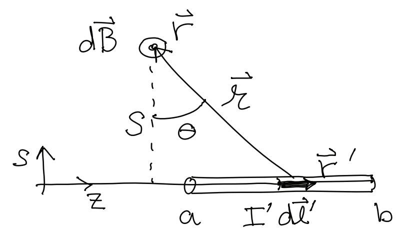


* Example 5.5: Parallel wires

$$\vec{B} = \frac{\mu_0}{4\pi} \int I' d\vec{l}' \times \frac{\vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{\mu_0 I'}{4\pi} \int_{z'=a}^{b} \frac{dz'\,\hat{z} \times (s\,\hat{s} + (z-z')\,\hat{z})}{(s^2 + (z-z')^2)^{3/2}}$$

$$= \frac{\mu_0 I'}{4\pi} \int_{z'=a}^{b} \frac{s\, dz'\, \hat{\phi}}{(s^2 + (z-z')^2)^{3/2}} = \frac{\mu_0 I'}{4\pi} \int \frac{s^2 \sec^2\theta\, d\theta\, \hat{\phi}}{\underbrace{(s^2(1+\tan^2\theta))^{3/2}}_{s^3 \sec^3\theta}}$$

$$= \frac{\mu_0 I'}{4\pi s} \int_{z'=a}^{b} \cos\theta\, d\theta\, \hat{\phi} = \frac{\mu_0 I'}{4\pi s} (\sin\theta_b - \sin\theta_a)\, \hat{\phi}$$



let $z - z' = -\tan\theta$

$dz' = s \sec^2\theta\, d\theta$

~ for an infinite wire: $\quad \vec{B} = \frac{\mu_0 I'}{2\pi s} \hat{\phi}$

~ for a second parallel wire: $\quad \vec{F} = \int_{\vec{r}} I d\vec{l} \times \vec{B} = -\frac{\mu_0}{2\pi} \frac{I I'}{s} \hat{s}\, l$

as shown before, this was unsed to define the Ampere (current) -> Tesla (B-field)

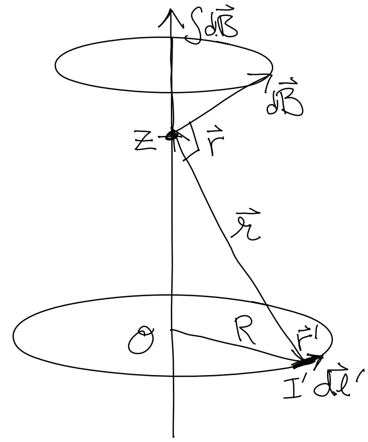
* Example 5.6: Current loop:

$$\vec{B} = \frac{\mu_0}{4\pi}\oint I'\vec{dl}' \times \frac{\vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{\mu_0}{4\pi}\int_{\phi'=0}^{2\pi} I' R\, d\phi' \hat{\phi} \times \frac{z\hat{z} - R\hat{s}'}{\mathscr{r}^3}$$

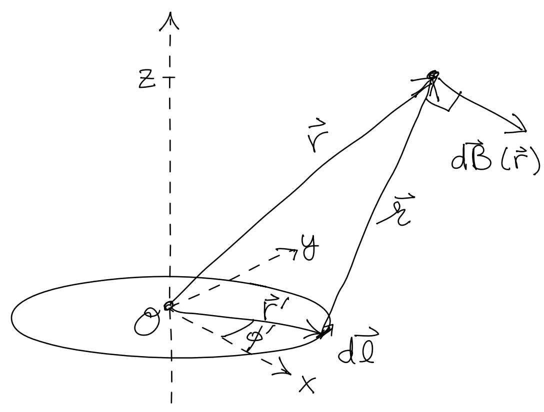
$$= \frac{\mu_0}{4\pi}\int_{\phi'=0}^{2\pi} I' R\, d\phi' \frac{z\hat{s}' + R\hat{z}}{\mathscr{r}^3}$$

$$= \frac{\mu_0}{4\pi}\int_{\phi'=0}^{2\pi} I' R\, d\phi' \frac{z(\cos\phi'\hat{x} + \sin\phi'\hat{y}) + R\hat{z}}{\mathscr{r}^3}$$

$$= \frac{\mu_0 I'}{4\pi} 2\pi \frac{R^2\hat{z}}{\mathscr{r}^3} = \frac{\mu_0 I' R^2 \hat{z}}{2(R^2+z^2)^{3/2}}$$



* Example: Off-axis field of current loop:



$$\vec{B} = \frac{\mu_0}{4\pi}\oint I'\vec{dl}' \frac{\vec{\mathscr{r}}}{\mathscr{r}^3}$$

$$= \frac{\mu_0}{4\pi}\int_{\phi'=0}^{2\pi} I' R\, d\phi' \hat{\phi}' \times \frac{z\hat{z} + (s\hat{s} - R\hat{s}')}{\left(z^2 + (s\hat{s} - R\hat{s}')^2\right)^{3/2}}$$

$$= \frac{\mu_0 I'}{4\pi}\int_{\phi'=0}^{2\pi} R\, d\phi' \hat{\phi}' \times \frac{z\hat{z} + s\hat{x} - R\hat{s}'}{(r^2 + R^2 - 2sR\cos\phi')^{3/2}}$$

$$= \frac{\mu_0 I'}{4\pi}\int_{\phi'=0}^{2\pi} R\, d\phi' (-\cancel{\sin\phi'}\hat{x} + \cos\phi'\hat{y}) \times \frac{(z\hat{z} + s\hat{x} + R\cancel{\sin\phi'}\hat{x} - R\cancel{\cos\phi'}\hat{y})}{(r^2 + R^2 - 2sR\cos\phi')^{3/2}}$$

$$= \frac{\mu_0 I' R}{4\pi}\int_{\phi'=0}^{2\pi} \frac{z\hat{x} - (s+R)\hat{z}\cos\phi'\, d\phi'}{(r^2 + R^2 - 2sR\cos\phi')^{3/2}}$$