Section 5.3 - Div and Curl of B

* the formalism of both electrostatics and magnetostatics follow the Helmholtz theorem
* these two diagrams illustrate the symmetry between the two forces

Coulomb & Superposition

$$\vec{E} = \int \frac{đq'\,\hat{\boldsymbol{\mathscr{r}}}}{4\pi\epsilon_0\,\mathscr{r}^2}$$

$$\vec{F} = q\vec{E} \qquad W = qV$$

Helmholtz

Integral field

$$\Phi_E = Q/\epsilon_0 \qquad \mathcal{E}_E = 0 \quad \text{(closed regions)}$$

Gauss / Stokes

Differential field

$$\nabla\cdot\vec{E} = \rho/\epsilon_0 \qquad \nabla\times\vec{E} = \mathbf{0}$$

Poincaré

$$\mathcal{E}_E = -\Delta V \quad \xleftrightarrow{\text{FTVC}} \quad \vec{E} = -\nabla V$$

Potential

$$V = \int \frac{đq'}{4\pi\epsilon_0\,\mathscr{r}}$$

Laplace / Green

Poisson

$$\nabla^2 V = -\rho/\epsilon_0$$

Biot-Savart & Superposition

$$\vec{B} = \int \frac{\mu 0}{4\pi}\frac{\vec{J}\,d\tau\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

$$\vec{F} = q\vec{v}\times\vec{B} \qquad \vec{p} = q\vec{A}$$

Helmholtz

Integral field

$$\mathcal{E}_B = \mu_0 I \qquad \Phi_B = 0 \quad \text{(closed regions)}$$

Stokes / Gauss

Differential field

$$\nabla\times\vec{B} = \mu_0\vec{J} \qquad \nabla\cdot\vec{B} = 0$$

? Poincaré

$$\xleftrightarrow{?} \quad \vec{B} = \nabla\times\vec{A}$$

Potential

$$\vec{A} = \int \frac{\mu_0}{4\pi}\frac{\vec{J}\,d\tau}{\mathscr{r}}$$

Laplace / Green

Poisson

$$\nabla^2\vec{A} = -\mu_0\vec{J}$$

* derivative chains

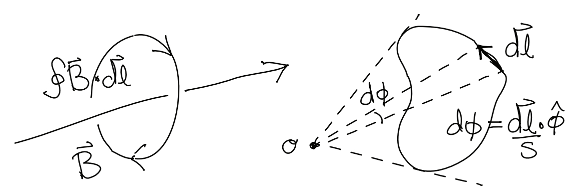
$$V \xrightarrow{-\nabla} \vec{E} \xrightarrow{\nabla\times} 0$$
$$\vec{E} \xrightarrow{\epsilon} \vec{D} \xrightarrow{\nabla\cdot} \rho$$

$$\chi \xrightarrow{\nabla} \vec{A} \xrightarrow{\nabla\times} \vec{B} \xrightarrow{\nabla\cdot} 0, \qquad \vec{A} \xrightarrow{\nabla\cdot} 0$$
$$\vec{H} \xrightarrow{\mu} \vec{B}$$
$$u \xrightarrow{-\nabla} \vec{H} \xrightarrow{\nabla\times} \vec{J} \xrightarrow{\nabla\cdot} 0$$

* integral equations

$$\mathcal{E}_B = \oint \vec{B}\cdot d\vec{l} = \mu_0 I \oint \frac{s\,d\phi}{2\pi s} = \mu_0 I \int_0^{2\pi} \frac{d\phi}{2\pi} = \mu_0 I$$

~ assumes exactly 1 winding, otherwise, $\mu_0 N I$



* differential equations (note: $đ\vec{q}' = \vec{J}'d\tau'$ or $\vec{K}'da'$ or $I'd\vec{l}'$)

$$\vec{B} = \int' \frac{\mu_0}{4\pi} đ\vec{q}' \times \nabla\frac{-1}{\mathscr{r}}$$
$$= \nabla\times\int' \frac{\mu_0}{4\pi}\frac{đ\vec{q}'}{\mathscr{r}}$$
$$= \nabla\times\vec{A} \qquad \vec{A} = \int' \frac{\mu_0\, đ\vec{q}'}{4\pi\mathscr{r}}$$

$$\nabla\cdot\vec{A} = \nabla\cdot\int' \frac{\mu_0\, đ\vec{q}'}{4\pi\mathscr{r}}$$
$$= \frac{\mu_0}{4\pi}\int' đ\vec{q}'\cdot\nabla\frac{1}{4\pi\mathscr{r}}$$
$$= \frac{\mu_0}{4\pi}\int' d\tau'\,\vec{J}'\cdot\nabla'\underbrace{\frac{-1}{4\pi\mathscr{r}}}_{G(\mathscr{r})} = 0$$

$$\int'\nabla'\cdot(\vec{J}'G)\,d\tau' = \int(\cancel{\nabla'\cdot\vec{J}'})G\,d\tau' + \int'\vec{J}'\cdot\nabla' G\,d\tau$$
$$= \oint d\vec{a}'\cdot(\vec{J}'G) = 0$$

$$\nabla\cdot\vec{B} = \nabla\cdot\nabla\times\vec{A} = 0 \quad \Longleftrightarrow$$

$$\Phi_B = \oint_{\partial S}\vec{B}\cdot d\vec{a} = \int_S \nabla\cdot\vec{B}\,d\tau = 0$$

$$\nabla\times\vec{B} = \nabla\times(\nabla\times\vec{A}) = \nabla(\overset{0}{\cancel{\nabla\cdot\vec{A}}}) - \nabla^2\vec{A} = \mu_0\vec{J}$$

$$-\nabla^2\vec{A} = -\nabla^2\int'\frac{\mu_0\, đ\vec{q}'}{4\pi\mathscr{r}} = \int'\mu_0 đ\vec{q}'\,\nabla^2\underbrace{\frac{-1}{4\pi\mathscr{r}}}_{G(\vec{\mathscr{r}})}$$
$$= \int'\mu_0\vec{J}(\vec{r}')\,\delta^3(\vec{r}-\vec{r}')\,d\tau' = \mu_0\vec{J}(\vec{r})$$

$$\mathcal{E}_B = \oint_{\partial S}\vec{B}\cdot d\vec{l} = \int_S \nabla\times\vec{B}\cdot d\vec{a} = \mu_0\int\vec{J}\cdot d\vec{a} = \mu_0 I_{enc}$$

note: $\nabla f(\mathscr{r}) = -\nabla' f(\mathscr{r})$

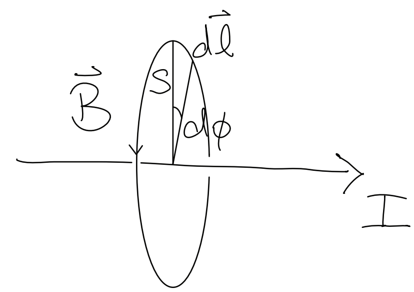
i.e. $\frac{\partial}{\partial x} f(x-x') = f'\cdot\frac{\partial(x-x')}{\partial x} = +f'$

$\frac{\partial}{\partial x'} f(x-x') = f'\cdot\frac{\partial(x-x')}{\partial x'} = -f'$

Applications of Ampere's law

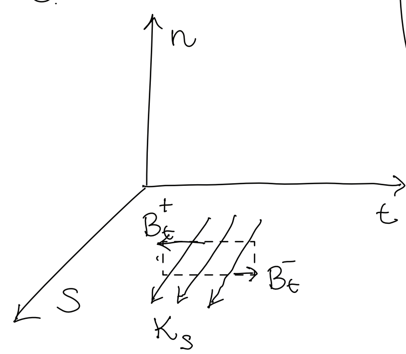
* Ampere's law is the analog of Gauss' law for magnetic fields
  - ~ uses a path integral around closed loop instead of integral over a closed surface
  - ~ simplest way to solve magnetic fields with high symmetry

* Example 5.7: straight wire



$$\oint \vec{B}\cdot d\vec{l} = B_\phi \cdot 2\pi s = \mu_0 I \qquad \vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

* Example 5.8: current septum



$$\oint \vec{B}\cdot d\vec{l} = (-B_t^+ + B_t^-)\, l = \mu_0 K_s l = \mu_0 I$$

$$\hat{n}\times\Delta\vec{B}_t = \vec{K} \quad \text{ie.} \quad B_t^{\pm} = \pm\frac{1}{2}K_s\hat{t}$$

* Example 5.9: infinite solenoid
  - ~ winding density n = #turns / length

$$K = N\frac{I}{l} = In$$

$$\oint \vec{B}\cdot d\vec{l} = (B_z - B_z)\cdot L = 0 \quad \text{outside.}$$

$$\oint \vec{B}\cdot d\vec{l} = (B_z - B_z)\cdot L = \mu_0 K L \qquad \Delta B = \mu_0 K \text{ again!}$$

* Maxwell's equations (steady-state E&M)

$$\nabla\cdot\vec{E} = \rho/\varepsilon_0 \qquad \nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{E} = 0 \qquad \nabla\times\vec{B} = \mu_0\vec{J}$$

$$\vec{F} = Q(\vec{E} + \vec{v}\times\vec{B})$$

  - ~ the two zeros mean there is no magnetic monopole
  - ~ actually as long as q_e/q_m is constant, a magnetic monopole can turned into an electric charge by a redefinition of E amd B (duality rotation)