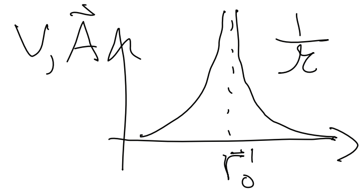
Section 5.4 - Magnetic Vector Potential

* Helmholtz theorem

$$\vec{B} = -\nabla\left(-\nabla^{-2}\,\cancel{\nabla\cdot\vec{B}}\right) + \nabla\times\overbrace{\left(-\nabla^{-2}\,\underbrace{\nabla\times\vec{B}}_{\mu_0 \vec{J}}\right)}^{\vec{A}} \qquad \nabla\cdot\vec{B} = 0 \ \rightleftharpoons\ \vec{B} = \nabla\times\vec{A}$$

$$\vec{A} = -\nabla^{-2}\mu_0\vec{J} = \frac{\mu_0}{4\pi}\int\frac{\vec{J}' d\tau'}{r}$$



$$\nabla\times\vec{B} = \nabla\times(\nabla\times\vec{A}) = \nabla\,\cancel{\nabla\cdot\vec{A}} - \nabla^2\vec{A} = \mu_0\vec{J}$$

* Gauge invariance: A is NOT unique! Only $\nabla\times\vec{A}$ specified, not $\nabla\cdot\vec{A}$ (Helmholtz)

$\nabla\times\nabla\lambda = 0$ so $\vec{A} = \vec{A}_0 + \nabla\lambda$ also satisfies $\vec{B} = \nabla\times\vec{A}$

~ λ is called a "gauge transformation", the set of all λ's forms a mathematical group
symmetry under gauge transformations is the basis of quantum field theories

~ a particular choice of A or a constraint on A is called a "gauge"

~ "Coulomb" or "radiation" gauge: $\nabla\cdot\vec{A} = 0$ always possible, unique up to B.C.'s
if $\vec{B} = \nabla\times\vec{A}_0$ let $\vec{A} = \vec{A}_0 + \nabla\lambda$ and solve for $\lambda \ni \nabla\cdot\vec{A} = 0$. (another Poission eq.)

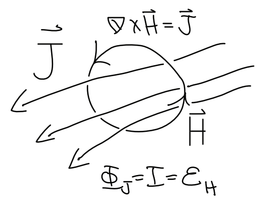
$$\nabla^2\lambda = -\nabla\cdot\vec{A}_0 \qquad \lambda = \frac{1}{4\pi}\int\frac{\nabla'\cdot\vec{A}_0'\,d\tau'}{r} = -\nabla^{-2}\,\nabla\cdot\vec{A}_0$$

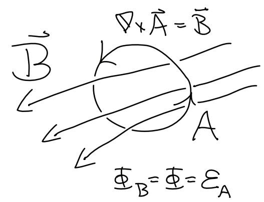
* Boundary conditions

H links current, A links flux

$$\mathcal{E}_H \equiv \oint_{\partial S}\vec{H}\cdot d\vec{l} = \int_S \nabla\times\vec{H}\cdot d\vec{a} = \int\vec{J}\cdot d\vec{a} = \Phi_J = I$$

$$\mathcal{E}_A \equiv \oint_{\partial S}\vec{A}\cdot d\vec{l} = \int_S \nabla\times\vec{A}\cdot d\vec{a} = \int_S\vec{B}\cdot d\vec{a} = \Phi_B$$





$$\left.\begin{array}{lll} \nabla\cdot\vec{A} = 0 & \Rightarrow \Phi_A = 0 & \hat{n}\cdot\Delta\vec{A} = 0 \\ \nabla\times\vec{A} = \vec{B} & \Rightarrow \mathcal{E}_A = \Phi_B & \hat{n}\times\Delta\vec{A} = \vec{0} \end{array}\right\} \quad \Delta\frac{\partial\vec{A}}{\partial t} = \Delta\frac{\partial\vec{A}}{\partial s} = 0 \qquad \Delta\vec{A} = \vec{0}$$

$$\left.\begin{array}{l} \nabla\cdot\vec{A} = 0 \Rightarrow \Delta\frac{\partial A_n}{\partial n} + \cancel{\Delta\frac{\partial A_t}{\partial t}} + \cancel{\frac{\partial A_s}{\partial s}} = 0 \qquad \Delta\frac{\partial A_n}{\partial n} = 0 \\ \hat{n}\times\Delta\vec{B} = \mu_0\vec{K} \Rightarrow \hat{n}\times\left(\hat{n}\,\partial_n + \cancel{\hat{s}\,\partial_s} + \cancel{\hat{t}\,\partial_t}\right)\times\Delta\vec{A} = -\Delta\frac{\partial\vec{A}_\perp}{\partial n} = \mu_0\vec{K}_\perp \end{array}\right\} \quad \Delta\frac{\partial\vec{A}}{\partial n} = -\mu_0\vec{K}$$

* Summary of vector potential

$$\begin{array}{ccccccc} \text{gauge} & & \text{potential} & & \text{field} & & \text{source} \\ \lambda & \xrightarrow{d} & (V,\vec{A}) & \xrightarrow{d} & (\vec{E},\vec{B}) & \xrightarrow{d} & 0 \\ & \text{gauge invariance} & & & \varepsilon \Big\Vert \mu \quad \text{Maxwell eq.'s} & & \\ & & & & (\vec{D},\vec{H}) & \xrightarrow{d} (\rho,\vec{J}) & \xrightarrow{d} 0 \end{array}$$

$(\vec{D},\vec{H}) \xrightarrow{d} (\rho,\vec{J})$: Poisson's eq.

$(\rho,\vec{J}) \xrightarrow{d} 0$: conservation of charge

$$\vec{A} \xrightarrow{\nabla\times} \vec{B} \xrightarrow{\nabla\times} \mu_0\vec{J}, \qquad \vec{A} \xrightarrow{-\nabla^2} \mu_0\vec{J}$$

Physical Significance of Vector Potential

* Physical significance: qV = potential energy $\quad qA$ = "potential momentum"
  - ~ it is the energy/momentum of interaction of a particle in the field
  - ~ some special cases can be solved using conservation of momentum, but you must account for momentum of the field unless there are no gradients
  - ~ $(V, \vec{A})$ is a 4-vector, like $(E, \vec{p})$ $\quad (c, \vec{v})$ $\quad (\rho, \vec{J})$
  - ~ $q(V - \vec{v}\cdot\vec{A})$ is a velocity-dependent potential

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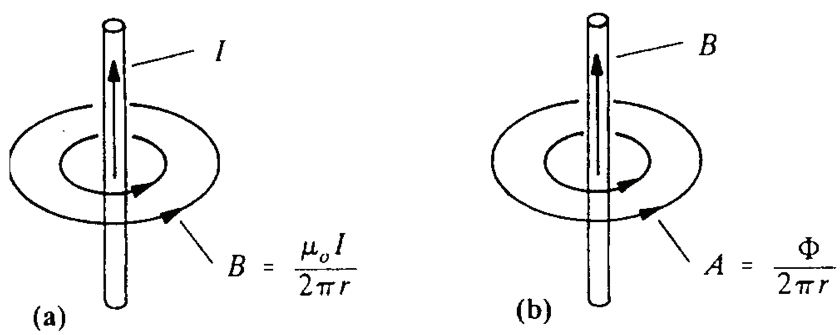
The vector **A** represents in direction and magnitude the time integral [that is, impulse] of the electromagnetic intensity which a particle placed at the point (x,y,z) would experience if the primary current were suddenly stopped.

[2]J. C. Maxwell, *A Treatise on Electricity and Magnetism* (Oxford University, Oxford, 1873), 1st ed. Article 590.

* B flux tube (solenoid)

$\nabla \times \vec{H} = \vec{J}$ $\qquad\qquad$ $\nabla \times \vec{A} = \vec{B}$

Wire	Solenoid	
$B = \frac{\mu_0 I}{2\pi s}$	$A = \frac{\Phi_B}{2\pi s}$	(inside)
$\vec{B} = \frac{\mu_0}{2}\vec{J}\times\vec{r}$	$\vec{A} = \frac{1}{2}\vec{B}\times\vec{r}$	(outside)



* Coaxial cable, straight conductor

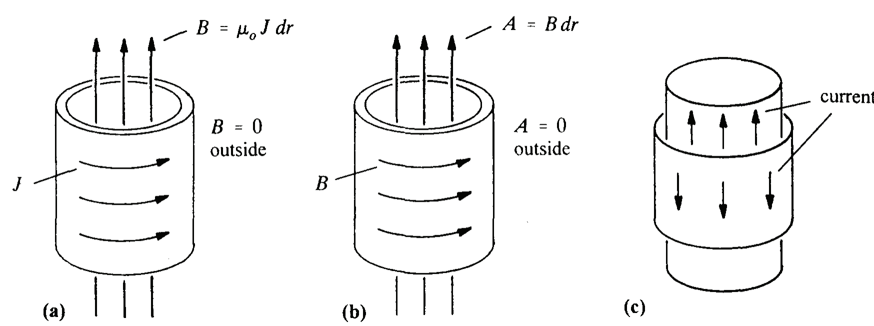
$$B = \mu_0 K = \mu_0 \frac{I}{2\pi s}$$

$$A = B\,dr = \frac{\mu_0 I\,ds}{2\pi s}$$

$$A(r) = \frac{\mu_0 I}{2\pi}\left[\ln(b) - \ln(s)\right]$$

$$\rightarrow \frac{-\mu_0 I}{2\pi}\ln(s)$$

$$V(r) = \frac{\lambda}{2\pi\epsilon_0}\ln(s)$$



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