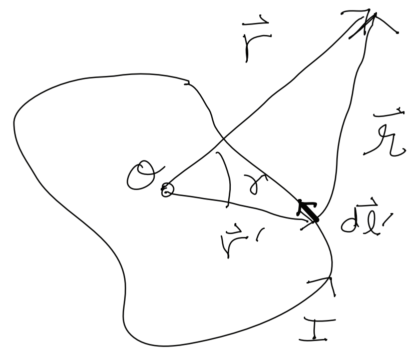
Section 5.4.3 - Multipole Expansion

* Similar to electrostatics, expand $1/r$



$$\vec A(\vec r) = \frac{\mu_0 I}{4\pi}\oint \frac{d\vec l'}{\mathscr{r}} = \frac{\mu_0 I}{4\pi}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\oint r'^{l} P_l(\cos\gamma)\, d\vec l'$$

$$\frac{1}{\mathscr{r}} = \frac{1}{\sqrt{r^2 - 2rr'\cos\gamma + r'^2}} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\gamma)$$

$$\vec A(\vec r) = \frac{\mu_0 I}{4\pi}\Big[\frac{1}{r}\oint d\vec l' + \frac{1}{r^2}\oint r'\cos\gamma\, d\vec l + \frac{1}{r^3}\oint r'^2\left(\frac{3}{2}\cos^2\gamma - \frac{1}{2}\right) d\vec l' + \dots\Big]$$

$$= \frac{\mu_0 I}{4\pi}\Big[\underbrace{\frac{1}{r}\cancel{\oint_{\text{loop}} d\vec l'}}_{\text{no monopole}} + \underbrace{\frac{\vec r}{r^3}\cdot\oint d\vec l'\,\vec r\,'}_{\text{dipole}} + \underbrace{\frac{\vec r}{r^5}\cdot\oint d\vec l'\left(\frac{3}{2}\vec r\,'\vec r\,' - \frac{1}{2}r'^2\right)\cdot\vec r}_{\text{quadrupole}}$$

$$\oint_{\partial S}\vec V\cdot d\vec l = \int_S \nabla\times\vec V\cdot d\vec a \qquad \text{(Stokes)}$$

divergence theorems

$$\boxed{\begin{aligned}\oint_{\partial S} d\vec l \cdots &= \int_S d\vec a\times\nabla \cdots\\ \oint_{\partial R} d\vec a \cdots &= \int_R d\tau\,\nabla \cdots\end{aligned}}$$

let $\vec V = \vec C T$ then $\nabla\times\vec V = \nabla\times\vec C T = \nabla T\times\vec C$

$$\vec c\cdot\oint_{\partial S} T\,d\vec l = \int_S \nabla T\times\vec c\cdot d\vec a = \vec c\cdot\int_S d\vec a\times\nabla T$$

$$\oint_{\partial S} T\,d\vec l = -\int_S \nabla T\times d\vec a$$

let $T = \vec c\cdot\vec r$ then $\displaystyle\oint_{\partial S}\vec c\cdot\vec r\,d\vec l = -\int_S \nabla(\vec c\cdot\vec r)\times d\vec a$

$$\oint_{\partial S}\vec c\cdot\vec r\,d\vec l = -\int\vec c\times d\vec a \qquad \vec c\times(\cancel{\nabla\times\vec r}) + \underbrace{(\vec c\cdot\nabla)\vec r}_{\vec c}$$

$$\oint \hat r\cdot\vec r\,'\,d\vec l' = -\hat r\times\int d\vec a'$$

$$\int_V \nabla T\,d\tau = \oint_{\partial V} d\vec a\, T \qquad \oint_{\partial V} d\vec a = 0 \text{ if } T=1 \quad \text{so} \quad \int_{S_1} d\vec a = \int_{S_2} d\vec a \text{ if } \partial S_1 = \partial S_2$$

$$\vec A_{dip}(\vec r) = \frac{\mu_0}{4\pi}\frac{\vec m\times\vec r}{r^3} \qquad \vec m \equiv \int I\,d\vec a = I\vec a$$

compare: $\times \leftrightarrow \cdot$ $\quad V_{dip}(\vec r) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\vec p\cdot\vec r}{r^3}$

* in spherical coordinates,

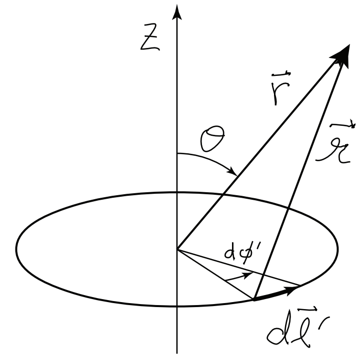
$$\vec B = \nabla\times\vec A = \frac{1}{r^2\sin\theta}\begin{vmatrix}\hat r & r\hat\theta & r\sin\theta\,\hat\phi\\ \partial_r & \partial_\theta & \partial_\phi\\ A_r & rA_\theta & r\sin\theta A_\phi\end{vmatrix} \qquad \vec A = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat\phi = A_\phi\hat\phi$$

$$= \frac{\mu_0 m}{4\pi}\frac{1}{r^2\sin\theta}\left[\hat r\,\partial_\theta\left(r\frac{\sin^2\theta}{r^2}\right) - r\hat\theta\,\partial_r\left(r\sin^2\theta\, A_\phi\right)\right]$$

$$\vec B_{dip}(\vec r) = \frac{\mu_0 m}{4\pi r^3}\left[2\cos\theta\,\hat r + \sin\theta\,\hat\theta\right]$$

compare: $\dfrac{p}{4\pi\varepsilon_0}\leftrightarrow\dfrac{\mu_0 m}{4\pi}$ $\quad \vec E = \dfrac{p}{4\pi\varepsilon_0 r^3}\left[2\cos\theta\,\hat r + \sin\theta\,\hat\theta\right]$

* *Example: current loop dipole*



$$\vec{\mathscr{r}} = (\vec{r}-\vec{r}\,') = (x\hat{x}+z\hat{z}) - (x'\hat{x}+y'\hat{y})$$

$$d\vec{\ell}\,' = d\vec{r}\,' = d(r'\cos\phi'\,\hat{x} + r'\sin\phi'\,\hat{y}) = r'd\phi'\hat{\phi}'$$

$$= -r'\sin\phi'\,\hat{x} + r'\cos\phi'\,\hat{y}\;\; d\phi' = (-y'\hat{x}+x'\hat{y})\,d\phi'$$

$$d\vec{\ell}\,'\times\vec{\mathscr{r}} = (-y'\hat{x}+x'\hat{y})\times(x\hat{x}+z\hat{z}-x'\hat{x}-y'\hat{y})\;d\phi'$$

$$= zy'\hat{y} + y'^2\hat{z} - x'x\hat{z} + x'z\hat{x} + x'^2\hat{z}\;d\phi'$$

$$= (zy'\hat{y} + r'^2\hat{z} + rx'\hat{\theta})\,d\phi'$$

$r\hat{\theta} = -x\hat{z} + z\hat{x}$ since $y=0$

$r'^2 = x'^2 + y'^2$

$$\vec{r}\cdot\vec{r}\,' = (x\hat{x}+z\hat{z})\cdot(x'\hat{x}+y'\hat{y}) = xx'$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\oint\frac{I'd\vec{\ell}\,'\times\vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{\mu_0 I'}{4\pi}\int_{\phi=0}^{2\pi}\frac{\cancel{zy'\hat{y}} + r'^2\hat{z} + rx'\hat{\theta}}{(r^2+xx'+r'^2)^{3/2}}\,d\phi'$$

~ *the above integral is antisymmetric under* $y \leftrightarrow y'$ *first term vanishes*

~ *to get the dipole approximation, assume* $r' \ll r$

$$\mathscr{r}^{-3} = \left(r^2\left(1+2\frac{xx'}{r^2}+\dots\right)\right)^{-3/2} = r^{-3}\left(1+3\frac{xx'}{r^2}+\dots\right)$$ *binomial expansion*

$$\vec{B}(\vec{r}) \simeq \frac{\mu_0 I'}{4\pi r^3}\int_{\phi=0}^{2\pi}(r'^2\hat{z} + rx'\hat{\theta})\left(1+3\frac{xx'}{r^2}\right)d\phi'$$ *order by powers of r' or x'*

$$= \frac{\mu_0 I}{4\pi r^3}\int_{\phi'=0}^{2\pi} rx'\hat{\theta} + \left(r'^2\hat{z} + 3\frac{xx'^2}{r}\hat{\theta}\right) + \mathcal{O}(r'^3)$$

~ *the first term* $= \int\cos\phi'\,d\phi' = 0$ — *no monopole!* $\int\cos^2\phi\,d\phi \sim \int\frac{1}{2}d\phi$

~ *the second two terms are the dipole* $\vec{m}\,' = I\vec{a} = (\pi r'^2)I'\hat{z}$

$$\vec{B}(\vec{r}) = \frac{\mu_0 I r'^2}{2r^4}\left(r\hat{z} + \frac{3}{2}x\hat{\theta}\right) = \frac{\mu_0}{4\pi}\,\frac{3\,\vec{m}\cdot\hat{r}\,\hat{r} - \vec{m}}{r^3}$$

$r\hat{z} = z\hat{r} - x\hat{\theta}$ since $y=0$

~ *equivalent to electric dipole under correspondence* $\dfrac{\vec{p}}{4\pi\varepsilon_0} \longleftrightarrow \dfrac{\mu_0\vec{m}}{4\pi}$