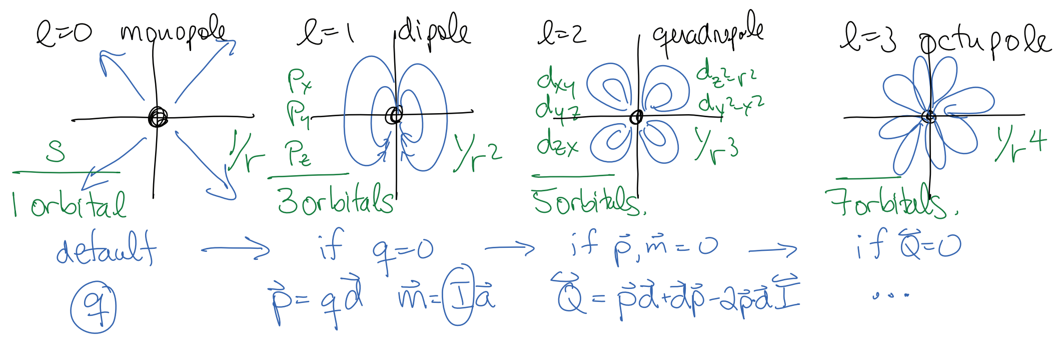
Section 6.3 – Auxiliary Field $\mathcal{H}$

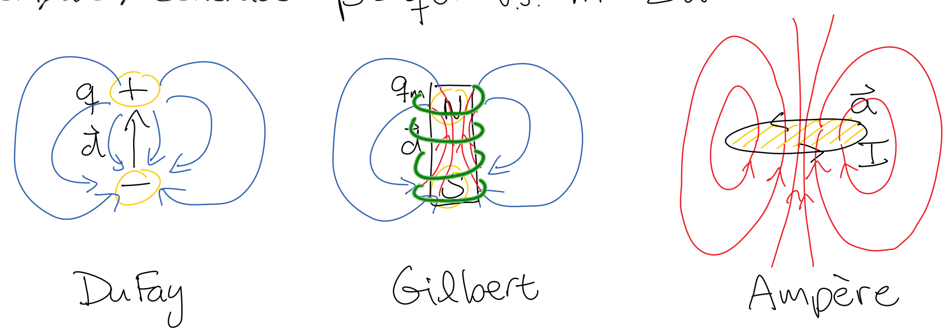
* reminder – Multipoles: gerneral solution to $\nabla^2 V = 0$ with azimuthal symmetry

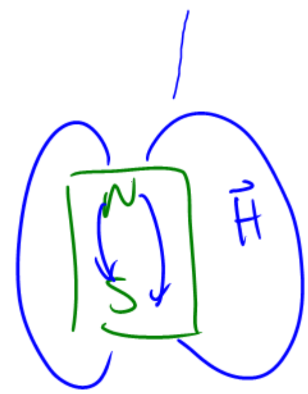
$$V = \frac{1}{4\pi\varepsilon_0} \sum_\ell \Big(\underbrace{Q_\ell^{ext} r^\ell}_{V_{int}} + \underbrace{Q_\ell^{int} r^{-\ell-1}}_{V_{ext}} \Big) P_\ell(c_\theta) \qquad \frac{1}{4\pi\varepsilon_0} \to \frac{\mu_0}{4\pi} \quad V \to U \qquad Q_\ell \to \begin{matrix} E_\ell \\ M_\ell \end{matrix} \text{ multipole moments.}$$

* magnetic multipoles



* compare / contrast $\vec{p} = q\vec{d}$ vs. $\vec{m} = I\vec{a}$





* polarization $\vec{P}$, magnetization $\vec{M}$ = dipole density

~ to get effective charge / current distribution:

a) expand V into dipole potential

b) integrate V due to dipole density field

c) $\hat{r}/r^2 = -\nabla \frac{1}{r}$, integrate by parts

d) compare with $V = \int \frac{\rho\, d\tau}{4\pi r} + \int \frac{\sigma\, da}{4\pi r}$ to get ρ_b, $\vec{J}_b$

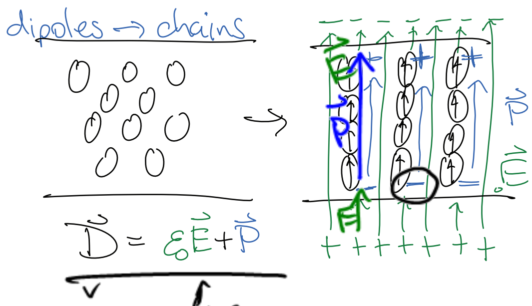
$$\boxed{\begin{matrix} V = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3} & \vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3} \\ \vec{p} \to d\vec{p} = \vec{P}\,d\tau & \vec{m} \to d\vec{m} = \vec{M}\,d\tau \\ \rho_b = -\nabla\cdot\vec{P} & \vec{J}_b = \nabla\times\vec{M} \end{matrix}}$$

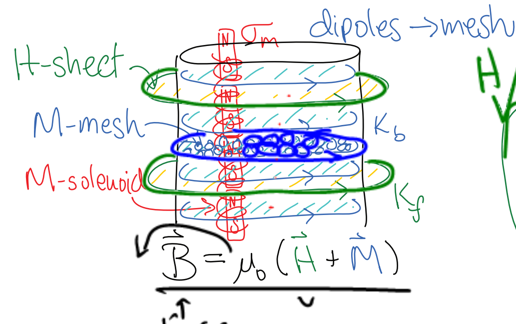
~ can also expand magnetic scalar potential to get "magnetic pole density" directly analogous with electric charge $\qquad \rho_m = -\nabla\cdot\vec{M}$

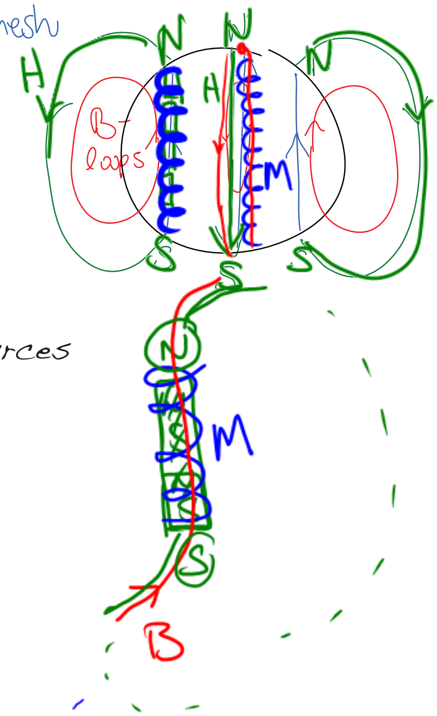
~ given permanent $\vec{P}$ or $\vec{M}$, use ρ_b, ρ_m or $\vec{J}_m$ to calculate fields

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$
$$\vec{B} = \mu_0 (\vec{H} + \vec{M})$$

* polarization chain; magnetization solenoid and meshets.







* source fields D, H only include free charge/current as sources

$$\nabla \cdot \epsilon_0 \vec{E} = \rho_f + \rho_b$$
$$\nabla \cdot \vec{P} = -\rho_b$$
$+$
$$\boxed{\begin{aligned} \nabla \cdot \vec{D} &= \rho_f \\ \nabla \times \vec{E} &= \vec{0} \end{aligned}}$$
$\rightarrow \sigma_b = -\hat{n}\cdot\vec{P}$

$$\nabla \times \vec{B}/\mu_0 = \vec{J}_f + \vec{J}_b$$
$$\nabla \times M = \vec{J}_b$$
$-$
$$\boxed{\begin{aligned} \nabla \times \vec{H} &= \vec{J}_f \\ \nabla \cdot \vec{B} &= 0 \end{aligned}} \quad \vec{M} = \chi_m \vec{H}$$
$\rightarrow \boxed{\begin{aligned} &\sigma_m = \hat{n}\cdot\vec{M} \\ &K_m = -\hat{n}\times\vec{M} \\ &\mu \end{aligned}}$

$$\vec{D} = \epsilon_0\vec{E} + \vec{P} = \epsilon\vec{E} \qquad \vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu\vec{H}$$
$$\epsilon = \epsilon_0\epsilon_r = \epsilon_0(1+\chi_e) \qquad \mu = \mu_0\mu_r = \mu_0(1+\chi_m)$$

* boundary conditions: $\nabla \rightarrow \hat{n}\Delta \quad d\tau \rightarrow da$
~ don't double count bound charge: K_b, σ_m, μ_r all account for the same thing!

fields	potentials	fields	potentials
$\hat{n}\times\Delta\vec{E} = 0$	$\Delta V = 0$	$\hat{n}\times\Delta\vec{H}_{(t)} = \vec{K}_{(s)}$	$\Delta U = -I \quad -\Delta\frac{\partial U}{\partial t} = K_s$
$\hat{n}\cdot\Delta\vec{D} = \sigma$	$-\Delta\epsilon\frac{\partial V}{\partial n} = \sigma$	$\hat{n}\cdot\Delta\vec{B} = \sigma_m$	$-\Delta\mu_r\frac{\partial U}{\partial n} = \sigma_m \quad \hat{n}\times\hat{t} = \hat{s}$

$\sigma_m = -\hat{n}\cdot\vec{M}$

* three ways to solve similar magnetic boundary value problems:
a) use Gilbert "pole density" ρ_m, σ_m explicitly ~ see Example #1
b) use Ampere "bound current" $\vec{J}_b$ explicitly ~ see Example #2
c) absorb magnetization into "permeability" μ ~ see Example #3

$K_b = \hat{n}\times\vec{M}$

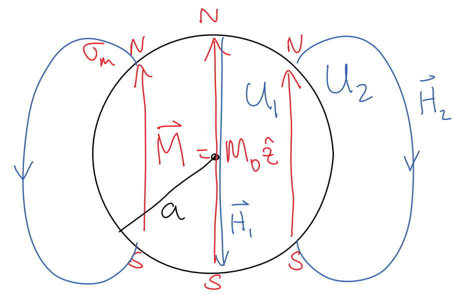
* *Example 1: Magnetic pole density σ_m*

$$\nabla \times \vec{H} = \vec{J}_f = \vec{0} \quad \rightarrow \quad H = -\nabla U$$

$$\nabla \cdot \vec{B} = \nabla \cdot \mu_0 (\vec{H} + \vec{M}) = 0 \quad \rightarrow \quad -\nabla^2 U = \rho_m \quad \text{no } \mu_0!$$

where $\rho_m = -\nabla \cdot \vec{M}$ $\qquad \sigma_m = \hat{n} \cdot \vec{M} = \underline{M_0 \cos\theta}$

B.C.'s: $\underline{U_1 = U_2 \qquad -\Delta \frac{\partial U}{\partial n} = \sigma_m}$



$$\vec{m} = \int \vec{M}\, d\tau = \tfrac{4}{3}\pi a^3 \vec{M}$$

$$U_1 = \sum_{\ell=0}^{\infty} \left[A_\ell (r/a)^\ell + \cancel{B_\ell} (a/r)^{\ell+1} \right] P_\ell (c_\theta)$$

$$U_2 = \sum_{\ell=0}^{\infty} \left[\cancel{C_\ell (r/a)^\ell} + D_\ell (a/r)^{\ell+1} \right] P_\ell (c_\theta)$$

BC #1: $U_1 \big|_{r=a} = U_2 \big|_{r=a} \quad \rightarrow \quad A_\ell = D_\ell$

BC #2: $$-\frac{\partial U_2}{\partial r}\Big|_a + \frac{\partial U_1}{\partial r}\Big|_a = \sum_\ell A_\ell \underbrace{\left[-(\ell+1)\left(\tfrac{-1}{a}\right) + (\ell)(a) \right]}_{2\ell+1/a} \overbrace{P_\ell (c_\theta)}^{\hat{e}_\ell} = M_0 \overbrace{P_1 (c_\theta)}^{\hat{e}_1}$$

$$A_0 = A_2 = A_3 = \dots = 0 \qquad A_1 = \tfrac{a}{3} M_0$$

$$U_1 = \tfrac{1}{3} M_0 r \cos\theta = \tfrac{1}{3} M_0 z \qquad U_2 = \frac{a^3}{3} M_0 \frac{\cos\theta}{r^2} = \frac{1}{4\pi} \frac{\vec{m}\cdot\vec{r}}{r^3} \quad \text{dipole}$$

$$\vec{H}_1 = -\vec{M}/3 \qquad \vec{H}_2 = \frac{1}{4\pi r^3}\left(3\hat{r}\,\hat{r}\cdot\vec{m} - \vec{m}\right) \equiv \frac{\mathbb{Q}\,\vec{m}}{4\pi r^3}$$

see probs: 3.33 p155, 5.33 p246

$$\vec{B}_1 = \mu_0 (\vec{H} + \vec{M}) = \tfrac{2}{3} \mu_0 \vec{M} \qquad \vec{B}_2 = \mu_0 \vec{H}_2$$

where $\mathbb{Q} \equiv 3\hat{r}\hat{r} \cdot - \mathbb{I}$

$\lim_{a \to 0}$: $$\boxed{\vec{E} = \frac{\mathbb{Q}\,\vec{p}}{4\pi\epsilon_0 r^3} - \frac{1}{3\epsilon_0} \vec{p}\, \delta^3(\vec{r})} \qquad \vec{H} = \frac{\mathbb{Q}\,\vec{m}}{4\pi r^3} - \frac{1}{3}\vec{m}\,\delta^3(\vec{r})$$

see prob 3.42 p157

if $\vec{m} = \int d\tau \vec{M}$ is fixed as $a \to 0$ then $\vec{M} \rightarrow \vec{m}\, \delta^3(\vec{r})$

$$\vec{D} = \frac{\mu_0 \mathbb{Q}\,\vec{p}}{4\pi r^3} + \frac{2}{3}\vec{p}\,\delta^3(\vec{r}) \qquad \boxed{\vec{B} = \frac{\mu_0 \mathbb{Q}\,\vec{m}}{4\pi r^3} + \frac{2}{3}\mu_0 \vec{m}\,\delta^3(\vec{r})}$$

see prob. 5.59 p253

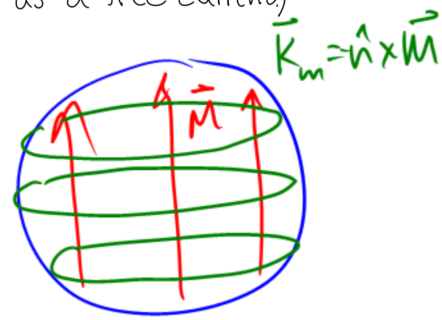
note closed $\vec{B} = \mu_0 (\vec{H} + \vec{M})$ lines of flux.

* Example 2: Bound current density $\vec{J}_b$

$\nabla\times\vec{H} = \vec{J}_{tot} = 0$ except on ∂ $\rightarrow$ $\vec{H} = -\nabla U$ discontinuity of U from

$\nabla\cdot\vec{B} = -\nabla\cdot\mu_0\nabla U = 0$ $\rightarrow$ $\nabla^2 U = 0$ K_b on the boundary

$\vec{J}_b = \nabla\times\vec{M} = 0$ $\underline{\vec{K}_b = -\hat{n}\times\vec{M}} = M_0\sin\theta\,\hat{\phi}$ (treat K_b as a free current)



BC's: $\Delta\frac{\partial U}{\partial n} = 0$ $-\Delta\frac{\partial U}{\partial t} = K_s$

$$U_1 = \sum_{l=0}^{\infty}\left[A_l\,(r/a)^l + \cancel{B_l\,(a/r)^{l+1}}\right]P_l(c_\theta)$$

$$U_2 = \sum_{l=0}^{\infty}\left[\cancel{C_l\,(r/a)^l} + D_l\,(a/r)^{l+1}\right]P_l(c_\theta)$$

BC#1: $$-\frac{\partial U_2}{\partial r}\Big|_a + \frac{\partial U_1}{\partial r}\Big|_a = \sum_{l=0}^{\infty}\left[-D_l\,(l+1)\left(\frac{-1}{a}\right) + A_l\,(l)\left(\frac{1}{a}\right)\right]P_l(c_\theta) = 0 \qquad D_l = \frac{-l}{l+1}A_l$$

BC#2: $$-\frac{\partial U_2}{r\partial\theta}\Big|_a + \frac{\partial U_1}{r\partial\theta}\Big|_a = \sum_{l=0}^{\infty}\left[-D_l + A_l\right]\frac{1}{a}P_l'(c_\theta)(-\sin\theta) = M_0\sin\theta$$

note: $P_l'(c_\theta)$ form a basis, and $P_1'(x) = 1$ like the RHS.

so $(-D_1 + A_1)\cdot\frac{-1}{a} = \left(\frac{-1}{2} - 1\right)A_1/a = M_0$ $\qquad A_1 = -\tfrac{2}{3}aM_0 = -2D_1$

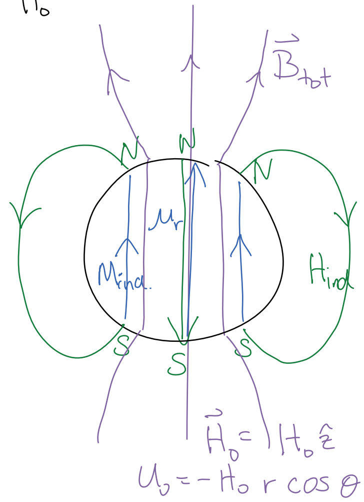
$U_1 = -\tfrac{2}{3}M_0 z$ $\qquad U_2 = \tfrac{1}{3}M_0\frac{\cos\theta}{r^2} = \frac{\vec{m}\cdot\vec{r}}{4\pi r^3}$

$\vec{B}_1 = \mu_0\vec{H}_1 = \mu_0\tfrac{2}{3}\vec{M}$ $\qquad \vec{B}_2 = \frac{\mathbb{Q}\,\vec{m}}{4\pi r^3}$

* note: $\vec{B} = \mu_0\vec{H}$, not $\mu_0(\vec{H}+\vec{M})$ inside because we replaced $\vec{M}$ with its effective current distribution K_b

* thus $\vec{H}$ is different from Ex#1, but $\vec{B}$ is still the same.

* Example 3: Magnetic permeability (linear homogeneous isotropic material)
Permeable sphere in an external constant field H_0



$\vec{M} = \chi_m \vec{H}$ encapsulated in $\vec{B} = \mu\vec{H}$ see Ex 4.7 p186

$\nabla\times\vec{H} = \vec{J}_{tot} = 0$ everywhere $\rightarrow$ $\vec{H} = -\nabla U$

$\nabla\cdot\vec{B} = -\nabla\cdot\mu\nabla U = 0$ $\rightarrow$ $\nabla^2 U = 0$

BC's: $U(\infty)\rightarrow -H_0 z$, $\Delta U = 0$, $\Delta\mu\frac{\partial U}{\partial n} = 0$

$$U_1 = \sum_{\ell=0}^{\infty}\left[A_\ell\,(r/a)^\ell + \cancel{B_\ell}\,(a/r)^{\ell+1}\right] P_\ell(c_\theta)$$

$$U_2 = \sum_{\ell=0}^{\infty}\left[C_\ell\,(r/a)^\ell + D_\ell\,(a/r)^{\ell+1}\right] P_\ell(c_\theta)$$

BC#0: $\lim_{r\to\infty} U_2(r) = \sum_{\ell=0}^{\infty} C_\ell (r/a)^\ell P_\ell(c_\theta) = -H_0\, r\, c_\theta$ $\qquad C_1 = -H_0 a$

we exclude all multipoles except $\ell=1$, since the source $\vec{H}_0$ is pure dipole.

BC#1: $U_2|_a - U_1|_a = (-H_0 a + D_1 - A_1) = 0$ $\quad\Rightarrow\quad D_1 = A_1 + H_0 a$

BC#2: $-\frac{\partial U_2}{\partial r}\Big|_a + \mu_r\frac{\partial U_1}{\partial r}\Big|_a = \left(H_0 + D_1\cdot 2/a + \mu_r A_1/a\right)\cos\theta = 0$

$3H_0 + (2+\mu_r)A_1/a = 0 \qquad A_1/a = \frac{-3}{2+\mu_r}H_0 \qquad D_1/a = \frac{-1+\mu_r}{2+\mu_r}H_0 = \frac{\chi_m}{2+\mu_r}H_0$

$U_1 = U_0 + \frac{\chi_m}{2+\mu_r}H_0 z = \frac{-3}{2+\mu_r}H_0 z \qquad \vec{H}_1 = \frac{3}{2+\mu_r}\vec{H}_0$

$U_2 = U_0 + \frac{\chi_m}{2+\mu_r}H_0 a^3\frac{\cos\theta}{r^2}$ where $U_0 = -H_0 z$

* compare with Ex#1: $U_1 = U_0 + \frac{1}{3}Mz \qquad U_2 = U_0 + \frac{1}{3}a^3 M\frac{\cos\theta}{r^2}$

$\vec{M} = \frac{3\chi_m}{2+\mu_r}\vec{H}_0 = \chi_m\left(\frac{3}{2+\mu_r}\right)\vec{H}_0 = \chi_m\vec{H}_1$ as dictated by $\vec{M} = \chi_m\vec{H}$

$\vec{B}_1 = \frac{3\mu}{2+\mu_r}\vec{H}_0 \qquad \vec{B}_2 = \mu_0\vec{H}_2$ $\qquad$ where $\vec{m} = \frac{4}{3}\pi a^3\cdot\frac{3\chi_m}{2+\mu_r}H_0$

$= \mu_0\left(\vec{H}_0 + \frac{\mathbb{Q}\vec{m}}{4\pi r^3}\right)$ $\qquad \mathbb{Q}\vec{m} = 3\hat{r}\hat{r}\cdot\vec{m} - \vec{m}$