Section 7.3, 10.1 - Maxwell's Equations

* towards a consistent system of field equations

gauge   potential   fields   sources

$$\Lambda \xrightarrow{d} (V,\vec{A}) \xrightarrow{d} (\vec{E},\vec{B}) \xrightarrow{d} 0$$

$$\varepsilon \updownarrow \mu$$

$$dd = 0 \qquad (\vec{D},\vec{H}) \xrightarrow{d} (\rho,\vec{J}) \xrightarrow{d} 0$$

Maxwell   continuity

$$\boxed{\begin{array}{ll} V \to V - \partial_t \Lambda & \vec{A} \to \vec{A} + \nabla\Lambda \\ \vec{E} = -\nabla V - \partial_t \vec{A} & \vec{B} = \nabla\times\vec{A} \\ \nabla\times\vec{E} + \partial_t \vec{B} = 0 & \nabla\cdot\vec{B} = 0 \\ \nabla\times\vec{H} - \partial_t \vec{D} = \vec{J} & \nabla\cdot\vec{D} = \rho \\ \partial_t \rho + \nabla\cdot\vec{J} = 0 & \vec{J} = \sigma\vec{E} \\ \vec{D} = \varepsilon\vec{E} = \varepsilon_0(\vec{E}) + \vec{P} & \vec{B} = \mu\vec{H} = \mu_0(\vec{H}+\vec{M}) \\ \vec{F} = q(\vec{E} + \vec{v}\times\vec{B}) = (\rho\vec{E} + \vec{J}\times\vec{B})\,d\tau & \end{array}}$$

* 2 problems
  a) potentials

$$\nabla\times\vec{E} = -\frac{\partial B}{\partial t} \quad \vec{E} \neq -\nabla V \implies \nabla\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = 0$$

$$\nabla\cdot\vec{B} = 0 \implies \vec{B} = \nabla\times\vec{A} \quad \vec{E} = -\nabla V - \partial_t \vec{A}$$

gauge invariance

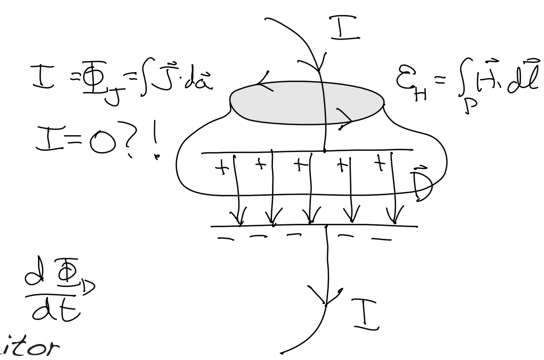
$$\begin{array}{l} V \to V - \partial_t \Lambda \\ \vec{A} \to \vec{A} + \nabla\Lambda \end{array} \implies \begin{array}{l} \vec{E} \to \vec{E} + (\cancel{\partial_t \nabla\Lambda} - \cancel{\partial_t \nabla\Lambda}) = 0 \\ \vec{B} \to \vec{B} + \cancel{\nabla\times(\nabla\Lambda)} \end{array}$$

  b) continuity

$$\nabla\cdot(\nabla\times E) = -\frac{\partial}{\partial t}\nabla\cdot B = 0$$

$$\nabla\cdot(\nabla\times\vec{H}) = \nabla\cdot\vec{J} \neq -\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial t}\nabla\cdot\vec{D} \implies \nabla\times\vec{H} = \vec{J} + \underbrace{\frac{\partial\vec{D}}{\partial t}}_{\text{"displacement current"}}$$

* example: capacitor - continuity: $I_{enc} = \frac{dQ}{dt}$
  ~ Ampere's law should not depend on surface to integrate charge flux
  ~ field should also exist in capacitor
  ~ each new charge on plate builds up a new D-flux line $\quad Q = \Phi_D$
  ~ charge "propagates" through capacitor via its associate D-flux line $\quad \Phi_J = I = \frac{dQ}{dt} = \frac{d\Phi_D}{dt}$
  ~ "displacement current": I flowing through wire = D building up in capacitor



* expand D, H

$$\nabla\cdot(\varepsilon_0\vec{E} + \vec{P}) = \rho_{tot} - \rho_b = \rho_f \qquad \vec{D} = \varepsilon_0\vec{E} + \vec{P} \quad \nabla\cdot\vec{P} = -\rho_b$$

$$\nabla\cdot\vec{B} = 0 \qquad \vec{B} = \mu_0(\vec{H} + \vec{M}) \quad \nabla\times\vec{M} = J_b$$

$$\nabla\times\vec{E} + \partial_t B = 0$$

$$\nabla\times\left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right) - \partial_t(\varepsilon_0\vec{E} + \vec{P}) = \vec{J}_f$$

$$\vec{J}_{tot} = \vec{J}_b + \underbrace{(\vec{J}_{d,tot} - \vec{J}_p)}_{\vec{J}_d} + \vec{J}_f$$

"displacement current"

$$\vec{J}_d \equiv \frac{\partial\vec{D}}{\partial t} = \varepsilon\frac{\partial\vec{E}}{\partial t}$$

* Maxwell's Eq's in vacuum

$$\nabla\cdot\vec{E}=\rho/\varepsilon_0 \qquad \nabla\times\vec{E}+\partial_t\vec{B}=0$$
$$\nabla\cdot\vec{B}=0 \qquad \nabla\times\vec{B}-\varepsilon_0\mu_0\partial_t\vec{E}=\mu_0\vec{J}$$

* integral form

$$\Phi_D=Q \qquad \mathcal{E}_E+\partial_t\Phi_B=0$$
$$\Phi_B=0 \qquad \mathcal{E}_H-\partial_t\Phi_D=I$$

* boundary conditions - integrate Maxwell's equations over the surface

$$\nabla\to\hat{n}\Delta \qquad \rho\to\sigma \qquad \vec{J}\to\vec{K} \qquad \int_{-\varepsilon}^{\varepsilon}dn\,\frac{\partial}{\partial n}=\Delta \qquad \int_{-\varepsilon}^{\varepsilon}dn\,\delta(n)=1$$

Fields

$$\hat{n}\cdot\Delta\vec{D}=\sigma \qquad \hat{n}\times\Delta\vec{E}=0$$
$$\hat{n}\cdot\Delta\vec{B}=0 \qquad \hat{n}\times\Delta\vec{H}=\vec{K}$$

Integral

$$\Delta\Phi_D=Q \qquad \Delta\mathcal{E}_E=-\Delta V=0$$
$$\Delta\Phi_B=0 \qquad \Delta\mathcal{E}_H=-\Delta U=I$$

Potentials

$$-\Delta\,\varepsilon\frac{\partial V}{\partial n}=\sigma \qquad -\Delta\frac{\partial V}{\partial t}=0$$
$$-\Delta\,\mu\frac{\partial U}{\partial n}=0 \qquad -\Delta\frac{\partial U}{\partial t}=K_s$$

* duality transformation - another symmetry of Maxwell's equations
  - ~ without sources, $B \Leftrightarrow E$ symmetry, except units $\quad c=\frac{1}{\sqrt{\varepsilon_0\mu_0}}$
  - ~ symmetry with sources by adding magnetic charge (monopole) ρ_m, $\vec{J}_m$
  - ~ single magnetic monopole in universe would imply quantization of charge $\quad \frac{\mu_0 q_e q_m}{4\pi}=\frac{n\hbar}{2}$
  - ~ magnetic contributions can be "rotated away" as long as q_e/q_m is constant

$$\nabla\cdot\vec{E}=\frac{1}{\varepsilon_0}\rho_e \qquad \nabla\times\vec{E}=-\mu_0\vec{J}_m-\frac{\partial B}{\partial t}$$
$$\nabla\cdot\vec{B}=\mu_0\rho_m \qquad \nabla\times\vec{B}=\mu_0\vec{J}_e+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$-\mu_0\left(\nabla\cdot\vec{J}_m+\frac{\partial\rho_m}{\partial t}\right)=0 \quad \text{(continuity)}$$
$$\mu_0\left(\nabla\cdot\vec{J}_e+\frac{\partial\rho_e}{\partial t}\right)=0$$