Electromagnetism in a Nutshell

* Maxwell's equations et al.

$$
\begin{array}{ccccccc}
\text{gauge} & & \text{potential} & & \text{field} & & \text{source} \\
\lambda & \xrightarrow{d} & (V,\vec{A}) & \xrightarrow{d} & (\vec{E},\vec{B}) & \xrightarrow{d} & 0 \\
\text{invariance} & & & & \varepsilon \int\!\!\int \mu & & \text{Maxwell eq.'s} \\
U & \xrightarrow{d} & (\vec{D},\vec{H}) & \xrightarrow{d} & (\rho,\vec{J}) & \xrightarrow{d} & 0
\end{array}
$$

Poisson   continuity

$$V \to V - \partial_t \Lambda \qquad \vec{A} \to \vec{A} + \nabla\Lambda$$

$$\vec{E} = -\nabla V - \partial_t \vec{A} \qquad \vec{B} = \nabla \times \vec{A}$$

$$\nabla \times \vec{E} + \partial_t \vec{B} = 0 \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{H} - \partial_t \vec{D} = \vec{J} \qquad \nabla \cdot \vec{D} = \rho$$

$$\partial_t \rho + \nabla \cdot \vec{J} = 0 \qquad \vec{J} = \sigma \vec{E}$$

$$\vec{D} = \varepsilon \vec{E} = \varepsilon_0 (\vec{E}) + \vec{P} \qquad \vec{B} = \mu \vec{H} = \mu_0 (\vec{H} + \vec{M})$$

$$d\vec{F} = dq(\vec{E} + \vec{v} \times \vec{B}) = (\rho \vec{E} + \vec{J} \times \vec{B})\, d\tau$$

* Flux and Flow

~ conserved currents

$$\Phi_J,\ \Phi_S,\ \Phi_T$$

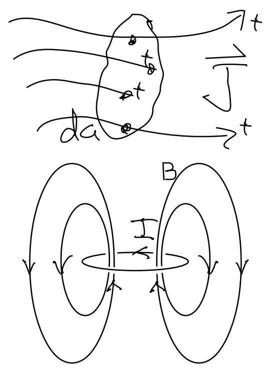
$$dq \sim \lambda dl \sim \sigma da \sim \rho d\tau$$

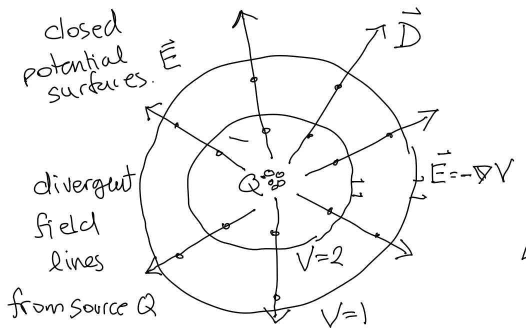
$$dq\,\vec{v} \sim I d\vec{l} \sim \vec{K} da \sim \vec{J} d\tau$$

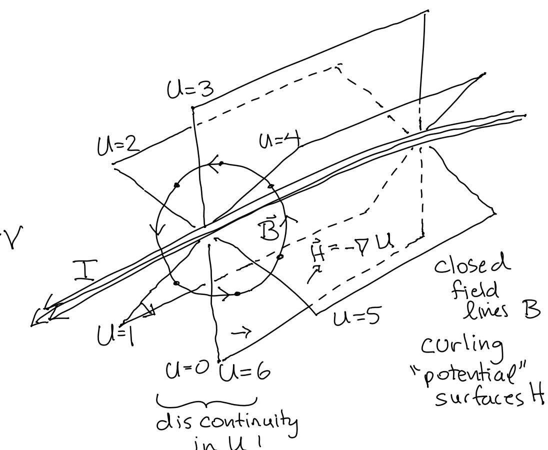
~ integral equations

$$\Phi_D = Q \qquad \mathcal{E}_E + \partial_t \Phi_B = 0$$

$$\Phi_B = 0 \qquad \mathcal{E}_H - \partial_t \Phi_D = I$$







* utility of D & B flux lines, E & H equipotential surfaces

~ flux through a surface $S = \Phi_B = \int_S \vec{B} \cdot d\vec{a}$ = # of lines that poke through a surface S

~ flow along a curve/path $P = \mathcal{E}_E = \int_P E \cdot d\vec{l}$ = # of surfaces that a path P pokes through

* potentials, from Helmholtz theorem, qV = potential energy $q\vec{A}$ = "potential momentum"

~ transverse and longitudinal components $n^2 \vec{v} = \vec{n}\,\hat{n} \cdot \vec{v} - \vec{n} \times \hat{n} \times \vec{v}$ $\nabla = \hat{n} \frac{\partial}{\partial n} + \nabla_t$

$$\vec{E} = -\vec{\nabla} \underbrace{\big(\vec{\nabla}^{-2} \overbrace{\nabla \cdot \vec{E}}^{\rho}\big)}_{} + \nabla \times \big(-\vec{\nabla}^{-2} \overbrace{\nabla \times \vec{E}}^{}\big) \qquad \vec{B} = -\vec{\nabla} \big(\vec{\nabla}^{-2} \overbrace{\nabla \cdot \vec{B}}^{0}\big) + \nabla \times \underbrace{\big(-\vec{\nabla}^{-2} \overbrace{\nabla \times \vec{B}}^{\vec{J}}\big)}_{}$$

$$= -\vec{\nabla}\ V \qquad \vec{B} = \nabla \times \vec{A}$$

* boundary conditions – integrate Maxwell's equations over the surface

$$\nabla \to \hat{n}\Delta \qquad \rho \to \sigma \qquad \vec{J} \to \vec{K} \qquad \int_{-\varepsilon}^{\varepsilon} dn \frac{\partial}{\partial n} = \Delta \qquad \int_{-\varepsilon}^{\varepsilon} dn\, \delta(n) = 1$$

~ electric

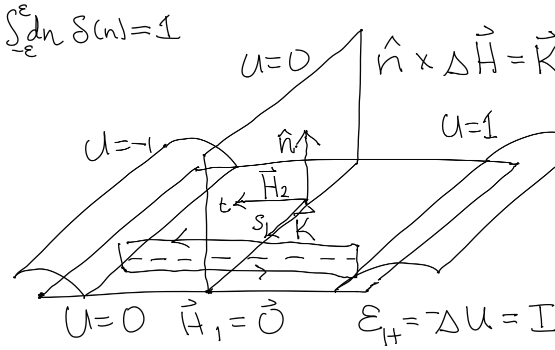
$$\mathcal{E}_E = 0 \qquad \Delta V = 0 \qquad E_{2t} = E_{1t} \qquad \hat{n} \times \Delta \vec{E} = 0$$

$$\Phi_D = Q \qquad -\Delta \varepsilon \frac{\partial V}{\partial n} = \sigma \qquad D_{2n} - D_{1n} = \sigma \qquad \hat{n} \cdot \Delta \vec{D} = \sigma$$

~ magnetic

$$\mathcal{E}_H = I \qquad -\Delta U = I \qquad H_{2t} - H_1 = K_s \qquad \hat{n} \times \Delta \vec{H} = \vec{K}$$

$$\Phi_B = 0 \qquad \Delta \mu \frac{\partial U}{\partial n} = 0 \qquad B_{2n} = B_{1n} \qquad \hat{n} \cdot \Delta \vec{B} = 0$$



~ surface current flows along U equipotential   ~ U is a SOURCE potential

~ the current $I = I_2 - I_1$ flows between any two equipotential lines $U = I_1$ and $U = I_2$

* electric magnetic dipoles and macroscopic equations (electric and magnetic materials)

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3} \qquad \vec{m} \equiv \oint I\, d\vec{a} = I\vec{a} \qquad \times \leftrightarrow \cdot \qquad V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3} \qquad \frac{\vec{p}}{4\pi\varepsilon_0} \leftrightarrow \frac{\mu_0\vec{m}}{4\pi} \qquad \vec{M} \equiv \frac{1}{\tau}\int_V d\tau\, \vec{m}$$

* dynamics of dipoles in fields (compare Electric and Magnetic)

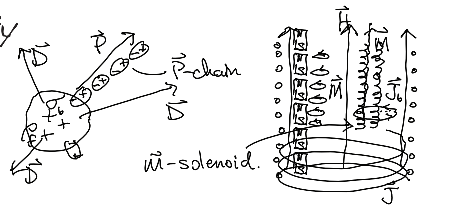
$$\vec{F}_e = (\vec{p}\cdot\nabla)\vec{E} = \nabla(\vec{p}\cdot\vec{E}) = -\nabla W \quad (\nabla\times\vec{E}=0) \qquad \vec{N}_e = \vec{p}\times\vec{E} \qquad W = -\int N\, d\theta = -\vec{p}\cdot\vec{E}$$

$$\vec{F}_m = (\vec{m}\times\nabla)\times\vec{B} = \nabla(\vec{m}\cdot\vec{B}) = -\nabla W \quad (\nabla\cdot\vec{B}=0) \qquad \vec{N}_m = \vec{m}\times\vec{B} \qquad W = -\int N\, d\theta = -\vec{m}\cdot\vec{B}$$

* constitutive relations: magnetic susceptibility and permeability

$$\varepsilon_0\vec{E} = \vec{D} - \vec{P} \qquad \vec{D} = \varepsilon_0(\vec{E}+\vec{P}) = \varepsilon_0(1+\chi_e)\vec{E} = \varepsilon_0\varepsilon_r\vec{E} = \varepsilon\vec{E}$$

$$\frac{1}{\mu_0}\vec{B} = \vec{H} + \vec{M} \qquad \vec{B} = \mu_0(\vec{H}+\vec{M}) = \mu_0(1+\chi_m)\vec{H} = \mu_0\underbrace{\mu_r}_{K_m}\vec{H} = \mu\vec{H}$$

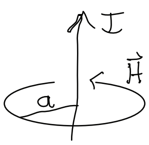


* three Ampere-like laws - each can be solved using Stoke's theorem

Ampere

$$\nabla\times\vec{H} = \vec{J}$$

$$\mathcal{E}_H = \Phi_J = I$$

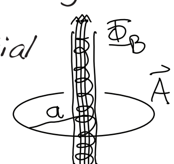


$$\vec{H} = \frac{I}{2\pi a}\hat{\phi}$$

Vector Potential

$$\nabla\times\vec{A} = \vec{B}$$

$$\mathcal{E}_A = \Phi_B$$

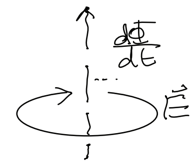


$$\vec{A} = \frac{\Phi_B}{2\pi a}\hat{\phi}$$

Faraday

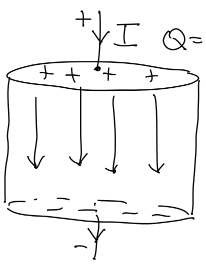
$$\nabla\times\vec{E} = \frac{\partial\vec{B}}{\partial t}$$

$$\mathcal{E}_E = \frac{-d\Phi_B}{dt}$$



$$\vec{E} = -\frac{d\Phi_B}{2\pi a\, dt}$$

* three passive electrical devices - each calculated by flux/flow = energy in field



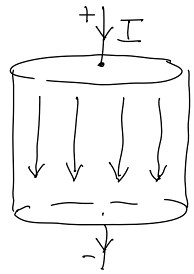
$$Q = \int I\,dt \qquad \vec{D} = \varepsilon\vec{E} \qquad \nabla\cdot\vec{D} = \rho$$

$$Q = \sigma_f A = \int\vec{D}\cdot d\vec{a} = \Phi_D$$

$$V = \int\vec{E}\cdot d\vec{a} = \mathcal{E}_E$$

$$C = Q/V = \varepsilon\,\Phi/\mathcal{E} = \varepsilon\frac{A}{l}$$

$$[C] = F \qquad [\varepsilon_0] = F/m$$



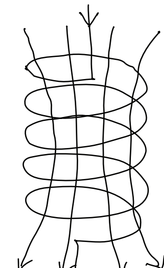
$$\vec{J} = \sigma\vec{E} \qquad \nabla\cdot J = \frac{\partial\rho}{\partial t}$$

$$I = \int\vec{J}\cdot d\vec{a} = \Phi_J$$

$$V = \text{same} = \mathcal{E}_E$$

$$R = V/I = \mathcal{E}/\sigma\Phi = \frac{l}{\sigma A}$$

$$[R] = \Omega \qquad [\sigma] = m/\Omega$$



$$\vec{B} = \mu\vec{H} \qquad \nabla\times\vec{H} = \vec{J}$$

$$NI = N\Phi_J = \mathcal{E}_H$$

$$V = -N\mathcal{E}_E = N\frac{d\Phi_B}{dt}$$

$$L = V/\dot{I} = \Phi/I = N^2\mu\frac{\Phi}{\mathcal{E}} = N^2\mu\frac{A}{l}$$

$$[L] = H \qquad [\mu_0] = H/m$$

* conserved currents

$$T_{\mu\nu} = \begin{array}{cc} \text{desity} & \text{flux} \\ \end{array}$$

$$T_{\mu\nu} = \begin{pmatrix} u & \vec{S} \\ \vec{\wp} & \overleftrightarrow{T} \end{pmatrix} \begin{array}{l} \text{energy} \\ \text{momentum} \end{array}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = \partial_\mu J^\mu = 0$$

$$\frac{\partial}{\partial t}(u_{mech} + u_{em}) + \nabla\cdot\vec{S} = 0 \qquad \vec{S} \equiv \vec{E}\times\vec{H}$$

$$\frac{\partial\vec{\wp}}{\partial t} + \nabla\cdot\overleftrightarrow{T} = 0 \qquad \overleftrightarrow{T} \equiv (\vec{D}\vec{E} + \vec{B}\vec{H}) - \tfrac{1}{2}(\vec{D}\cdot\vec{E} + \vec{B}\cdot\vec{H})\overleftrightarrow{I}$$

* wave equations (Helmholtz) and solutions (Green's functions)

$$\left(\varepsilon\mu\frac{\partial^2}{\partial t^2} - \nabla^2\right)V = \rho/\varepsilon \qquad \left(\varepsilon\mu\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{A} = \mu\vec{J}$$

$$\left(\varepsilon\mu\frac{\partial^2}{\partial t^2} - \nabla^2\right)\lambda = 0$$

$$\left(\varepsilon\mu\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{E} = 0$$

$$\left(\varepsilon\mu\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{B} = 0$$

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r}',t_r)\,d\tau'}{\mathscr{r}} \qquad A(\vec{r},t) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}',t_r)\,d\tau}{\mathscr{r}}$$

* application of oblique boundary conditions: Fresnel equations

i) $D_{1n} = D_{2n}$:
ii) $B_{1n} = B_{2n}$:
iii) $E_{1t} = E_{2t}$:
iv) $H_{1t} = H_{2t}$:

$$E_I - E_R = \beta E_T \qquad E_R = \frac{\alpha-\beta}{\alpha+\beta}E_I \qquad R = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2$$

$$E_I + E_R = \alpha E_T \qquad E_T = \frac{2}{\alpha+\beta}E_I \qquad T = \frac{4\alpha\beta}{(\alpha+\beta)^2}$$

$$I = \tfrac{1}{2}\varepsilon v E^2\cos\theta = \tfrac{1}{2}\kappa\cos\theta\, E^2$$

$$\alpha = \frac{\cos\theta_2}{\cos\theta_1} \qquad \beta = \frac{\kappa_2}{\kappa_1}$$

* guides: wave equation for longitudinal component, boundary conditions

$$\left[\nabla_t^2 + (\omega/c)^2 - k^2\right]\begin{Bmatrix} E_z \\ B_z \end{Bmatrix} = 0.$$

$$(TE)\quad B_z(x,y)e^{ikz} \qquad \left.\frac{\partial B_z}{\partial n}\right|_s = 0 \qquad E_z = 0$$

$$(TM)\quad E_z(x,y)e^{ikz} \qquad E_z\big|_s = 0 \qquad B_z = 0$$