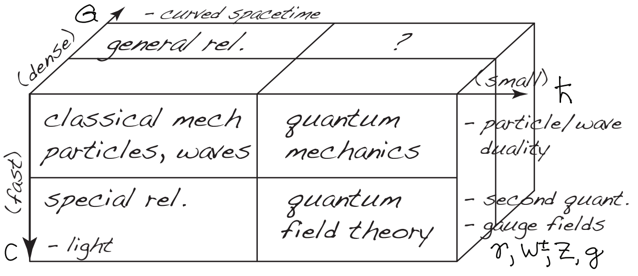
Survey of Electromagnetism

* Realms of Mechanics



~ E&M was second step in unification

~ the stimulus for special relativity

~ the foundation of QED -> standard model

* Electric charge (duFay, Franklin)

~ +,- equal & opposite (QCD: r+g+b=0)

~ $e = 1.6\times10^{-19}$ C, quantized ($q_n < 2\times10^{-21}$ e)

~ locally conserved (continuity)

* Electric Force (Coulomb, Cavendish)

$$\vec{F} = \underbrace{\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}}_{\vec{E}}\cdot q$$

* Electric Field (Faraday)

~ action at a distance vs. locality
field "mediates" or carries force
extends to quantum field theories

~ field is everywhere always $\vec{E}(x,t)$
differentiable, integrable
field lines, equipotentials

~ powerful techniques
for solving complex problems

* Field lines / Flux

~ E is tangent to the field lines
Flux = # of field lines

~ density of the lines = field strength
D is called "electric flux density"

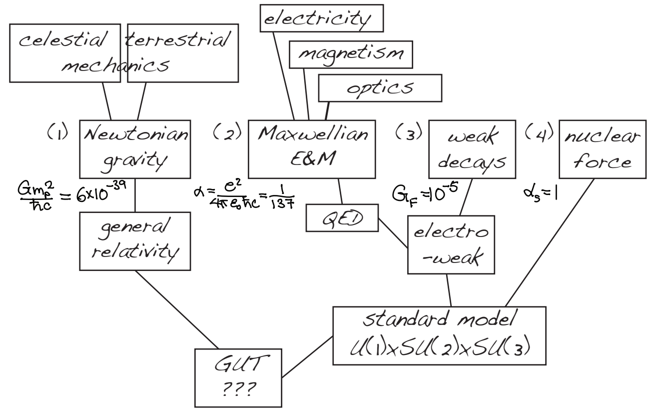
~ note: $\frac{A}{r^2} = \Omega$ independent of distance

$$\boxed{\Phi_D = \int D\cdot d\vec{a}}$$

$$\vec{D} = \varepsilon\vec{E} = \Phi_D/A$$

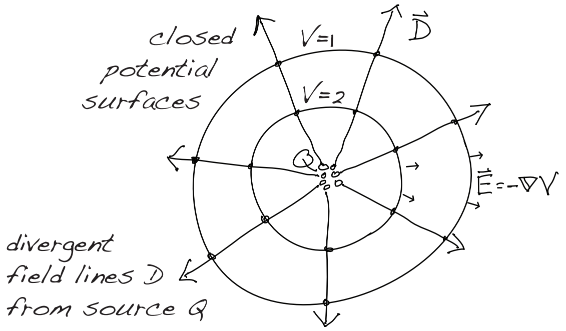
electric flux
flows from
(+) ⟶ (−)
all flux lines begin at +
and end at − charge

* Unification of Forces



* Electric potential

$\underset{\text{force}}{\vec{F}} = q\underbrace{\vec{E}}_{\text{field}}$	$\vec{F} = m\underbrace{\vec{g}}_{\text{grav. field}}$
$\underset{\text{energy}}{U} = q\underbrace{Ed}_{\text{potential}}$	$U = m\underbrace{gh}_{\text{"danger"}}$



* Equipotential surfaces / Flow

~ no work done to field lines
Equipotentials = surfaces of const energy

~ work is done along field line
Flow = # of potential surfaces crossed

$$\boxed{\mathcal{E}_E \equiv \int \vec{E}\cdot d\vec{\ell}}$$

$$V = -\mathcal{E}_E$$

$$E = -\nabla V$$

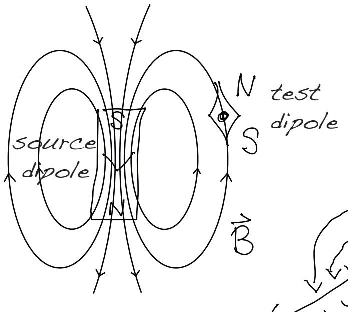
~ potential if flow
is independent of path

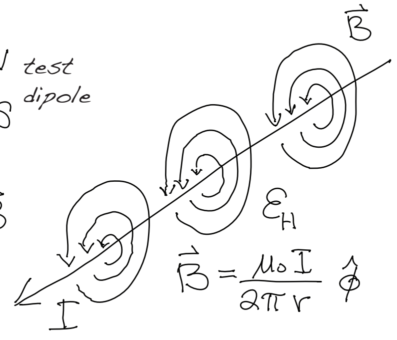
~ circulation or EMF in a closed loop

* Magnetic field

- ~ no magnetic charge (monopole)
- ~ field lines must form loops
- ~ permanent magnetic dipoles first discovered

torque: $\vec{\tau} = \vec{\mu} \times \vec{B}$
energy: $U = -\vec{\mu} \cdot \vec{B}$
force: $\vec{F} = \nabla(\vec{\mu} \cdot \vec{B})$



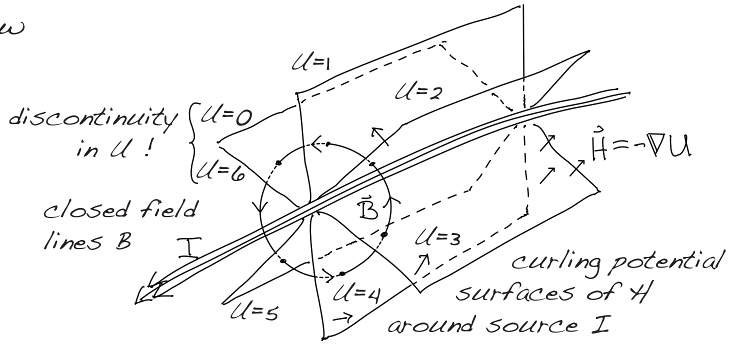


$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$$

- ~ electric current shown to generate fields (Oersted, Ampere)
- ~ magnetic dipoles are current loops
- ~ Biot-Savart law - analog of Coulomb law

$$\vec{F} = \int I' d\vec{l} \times \underbrace{\frac{\mu_0}{4\pi}\int \frac{I d\vec{l} \times \hat{r}}{r^2}}_{\vec{B}}$$

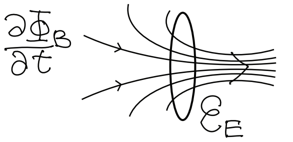
- ~ B = flux density
- ~ H = field intensity $\quad \vec{B} = \mu\vec{H} = \Phi_B/A$



* Faraday law

- ~ opposite of Orsted's discovery:
  changing magnetic flux induces potential (EMF)
- ~ electric generators, transformers

$$\mathcal{E}_E = -\frac{\partial \Phi_B}{\partial t}$$



* Maxwell equations

- ~ added displacement current - D lines have +/- charge at each end
- ~ changing diplacement current equivalent to moving charge
- ~ derived conservation of charge and restored symmetry in equations
- ~ predicted electromagnetic radiation at the speed of light $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

$$+ \xrightarrow{\Phi_D} -$$

$$I_d = \frac{\partial \Phi_D}{\partial t}$$

Maxwell equations

$$\nabla \cdot \vec{D} = \rho \qquad \nabla \times \vec{E} + \partial_t \vec{B} = \vec{0}$$
$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{H} - \partial_t \vec{D} = \vec{J}$$

$$\boxed{\begin{array}{ll} \Phi_D = Q_{encl} & \Phi_B = 0 \\ \mathcal{E}_E = -\frac{\partial \Phi_B}{\partial t} & \mathcal{E}_H = I_{encl} + \frac{\partial \Phi_D}{\partial t} \end{array}}$$

Constitutive equations

$$\vec{D} = \varepsilon\vec{E} \quad \vec{B} = \mu\vec{H} \quad \vec{J} = \sigma\vec{E}$$

Lorentz force

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) = \int(\rho\vec{E} + \vec{J} \times \vec{B})$$

$$\begin{tikzcd}
 & (0) & (1) & (2) & (3) & (4) \\
0 \arrow[r] & \lambda \arrow[r, "d"] & (V,\vec{A}) \arrow[r, "d"] & (\vec{E},\vec{B}) \arrow[r, "d"] \arrow[d, shift right, "\varepsilon"'] \arrow[dr, "\sigma"] & 0 & \\
 & & (\ ,U) \arrow[r, dash, "d"] & (\vec{D},\vec{H}) \arrow[r, "d"] \arrow[u, shift right, "\mu"'] & (\rho,\vec{J}) \arrow[r, "d"] & 0
\end{tikzcd}$$

Continuity

$$\nabla \cdot \vec{J} + \partial_t \rho = 0$$

Potentials

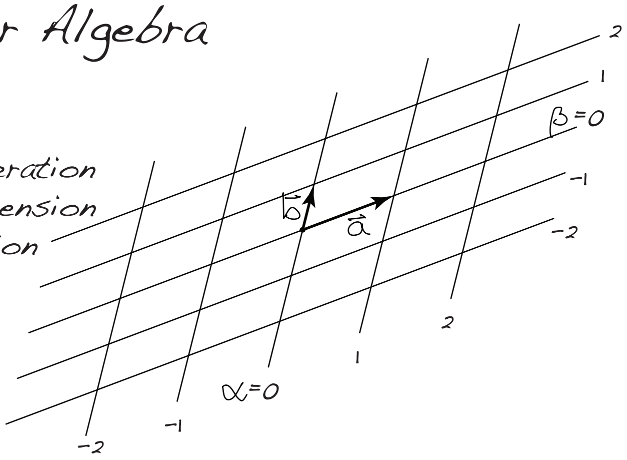
$$\vec{E} = -\nabla V - \partial_t \vec{A} \quad \vec{B} = \nabla \times \vec{A}$$

Gauge transformation

$$V \to V - \partial_t \lambda \quad \vec{A} \to \vec{A} + \nabla\lambda$$

Wave equation $\quad -\Box^2(V,\vec{A}) = (\rho/\varepsilon, \mu\vec{J})$

Section 1.1 - Vector Algebra



* Linear (vector) space

~ linear combination: $(\alpha\vec{u}+\beta\vec{v})$ is the basic operation

~ basis: ($\hat{x},\hat{y},\hat{z}$ or $\vec{a},\vec{b},\vec{c}$) # basis elements = dimension
independence: not collapsed into lower dimension
closure: vectors span the entire space

~ components: $\vec{x} = \vec{a}\alpha+\vec{b}\beta+\vec{c}\gamma = (\vec{a}\,\vec{b}\,\vec{c})\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$
in matrix form:

$$\vec{x} = \vec{\mathbb{b}}\,\mathbb{X}$$

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}a_x & b_x & c_x\\ a_y & b_y & c_y\\ a_z & b_z & c_z\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$$

where

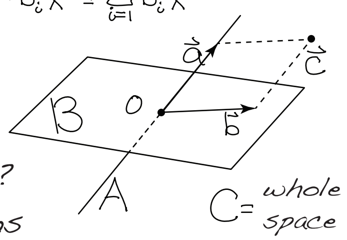
$$\vec{a}=\hat{x}a_x+\hat{y}a_y+\hat{z}a_z=(\hat{x}\,\hat{y}\,\hat{z})\begin{pmatrix}a_x\\ a_y\\ a_z\end{pmatrix}, \text{etc}$$

(usually one upper, one lower index)

~ Einstein notation: implicit summation over repeated indices $\vec{x}=\vec{b}_i x^i \equiv \sum_{i=1}^{3}\vec{b}_i x^i$

~ direct sum: $C=A\oplus B$ add one vector from each independent space to get vector in the product space (not simply union)

~ projection: the vector $\vec{c}=\vec{a}+\vec{b}$ has a unique decomposition ('coordinates' $(\vec{a},\vec{b})$ in A, B) - relation to basis/components?

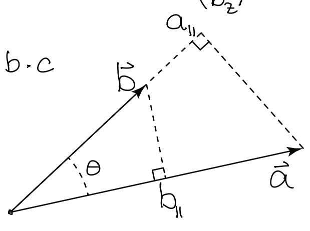


~ all other structure is added on as multilinear (tensor) extensions

* Metric (inner or dot product, contraction) - distance and angle - reduces dimension

$$c=\vec{a}\cdot\vec{b}=ab\cos\theta=a_{\parallel}b=ab_{\parallel}=a_xb_x+a_yb_y+a_zb_z=a_ib^i=(a_x a_y a_z)\begin{pmatrix}b_x\\ b_y\\ b_z\end{pmatrix}$$

~ properties: 1) scalar valued - what is outer product?
2) bilinear form $a\cdot(b+c)=a\cdot b+a\cdot c$ $(a+b)\cdot c=a\cdot c+b\cdot c$
3) symmetric $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$



~ orthonormality and completeness - two fundamental identities help to calculate components, implicitly in above formulas

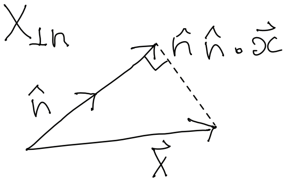
$$\boxed{\begin{aligned}\hat{e}_i\cdot\hat{e}_j &= \delta_{ij}\\ \sum_{i=1}^{3}\hat{e}_i\hat{e}_i\cdot &= I\end{aligned}}$$

Kroneker delta: components of the identity matrix

$$\delta_{ij}=\begin{cases}1 \text{ if } i=j\\ 0 \text{ if } i\neq j\end{cases}\sim\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=I$$

$$a^i=\vec{a}\cdot\hat{e}^i=a^1\hat{e}_1\cdot\hat{e}^i+a^2\hat{e}_2\cdot\hat{e}^i+a^3\hat{e}_3\cdot\hat{e}^i$$

~ orthogonal projection: a vector $\vec{n}$ divides the space X into $X_{\parallel n}\oplus X_{\perp n}$
geometric view: dot product $\hat{n}\cdot\vec{x}$ is length of $\vec{x}$ along $\hat{n}$
Projection operator: $P_{\parallel}\equiv\hat{n}\hat{n}\cdot$ acts on x: $P_{\parallel}\vec{x}=\vec{x}_{\parallel}=\hat{n}\,\hat{n}\cdot\vec{x}$



~ generalized metric: for basis vectors which are not orthonormal, collect all nxn dot products into a symmetric matrix (metric tensor)

$$g_{ij}=\vec{b}_i\cdot\vec{b}_j \qquad \vec{x}\cdot\vec{y}=x^i\vec{b}_i\cdot\vec{b}_jy^j=x^ig_{ij}y^j$$
$$=\mathbb{X}^T\vec{\mathbb{b}}^T\cdot\mathbb{b}\mathbb{Y}=\mathbb{X}^Tg\,\mathbb{Y}=(x^1x^2x^3)\overbrace{\begin{pmatrix}g_{11}&g_{12}&g_{13}\\ g_{21}&g_{22}&g_{23}\\ g_{31}&g_{32}&g_{33}\end{pmatrix}}^{g}\begin{pmatrix}y^1\\ y^2\\ y^3\end{pmatrix}$$

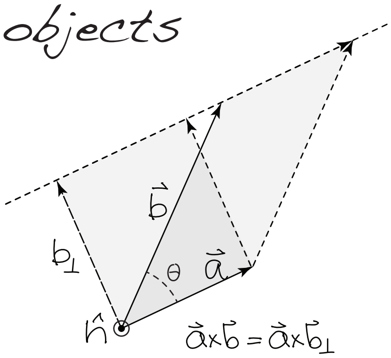
in the case of a non-orthonormal basis, it is more difficult to find components of a vector, but it can be accomplished using the reciprocal basis (see HW1)

Exterior Products - higher-dimensional objects

* cross product (wedge product, area)

$$\vec{c} = \vec{a}\times\vec{b} = \hat{n}\,ab\sin\theta = \hat{n}\,a_\perp b = \hat{n}\,ab_\perp = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

where $\hat{n}\perp\vec{a}$ and $\hat{n}\perp\vec{b}$ (RH-rule)



$\vec{a}\times\vec{b} = \vec{a}\times\vec{b}_\perp$

$\vec{a}\times(\vec{b}-\vec{b}_\perp) = \vec{0}$

(parallel)

~ properties:
1) vector-valued
2) bilinear $\quad a\times(b+c) = a\times b + a\times c \quad (a+b)\times c = a\times c + b\times c$
3) antisymmetric $\quad \vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$ (oriented)

~ components: $\hat{e}_i\times\hat{e}_j = \varepsilon_{ij}{}^{k}\,\hat{e}_k$

where $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$
$\varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321} = -1$

Levi-Civita tensor - completely antisymmetric:

$$\varepsilon_{ijk} = \begin{cases} 1 & ijk \text{ even permutation} \\ -1 & ijk \text{ odd permutation} \\ 0 & \text{repeated index} \end{cases}$$

(ijk cyclic)

$$\vec{x}\times\vec{y} = x^i\hat{e}_i\times\hat{e}_j y^j = \varepsilon_{ij}{}^{k}\,x^i y^j\,\hat{e}_k$$

~ orthogonal projection: $\hat{n}\times$ projects ⊥ to $\hat{n}$ and rotates by 90°

$$\hat{x}_\perp = -\hat{n}\times(\hat{n}\times\hat{x}) = P_\perp\vec{x} \qquad P_\perp = -\hat{n}\times\hat{n}\times \qquad \boxed{P_\parallel + P_\perp = \hat{n}\hat{n}\cdot - \hat{n}\times\hat{n}\times = I}$$

~ where is the metric in x?
vector x vector = pseudovector
symmetries act more like a 'bivector'
can be defined without metric

* triple product (volume of parallelpiped) - base times height
$$d = \vec{a}\cdot\vec{b}\times\vec{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$
~ completely antisymmetric - definition of determinant
~ why is the scalar product symmetric / vector product antisymmetric?
~ vector · vector x vector = pseudoscalar (transformation properties)
~ acts more like a 'trivector' (volume element)
~ again, where is the metric? (not needed!)

* exterior algebra (Grassman, Hamilton, Clifford)
~ extended vector space with basis elements from objects of each dimension
~ pseudo-vectors, scalar separated from normal vectors, scalar

magnitude,	length,	area,	volume
scalar,	vectors,	bivectors,	trivector
1,	$\hat{x},\hat{y},\hat{z}$,	$\hat{x}\hat{y},\hat{y}\hat{z},\hat{z}\hat{x}$,	$\hat{x}\hat{y}\hat{z}$

~ what about higher-dimensional spaces (like space-time)?
can't form a vector 'cross-product' like in 3-d, but still have exterior product

~ all other products can be broken down into these 8 elements
most important example: BAC-CAB rule (HW1: relation to projectors)

$$A\times(B\times C) = B(A\cdot C) - C(A\cdot B)$$

$$\varepsilon^{i}{}_{jk}A^j(\varepsilon^{k}{}_{mn}B^m C^n) = (\delta^i_m\delta_{jn} - \delta^i_n\delta_{jm})A^j B^m C^n = B^i(A^j C_j) - C^i(A^j B_j)$$

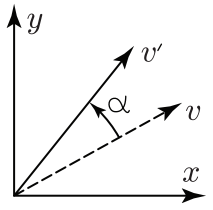
Section 1.1.5 - Linear Operators

* Linear Transformation
- ~ function which preserves linear combinations
- ~ determined by action on basis vectors (egg-crate)
- ~ rows of matrix are the image of basis vectors
- ~ determinant = expansion volume (triple product)
- ~ multilinear (2 sets of bases) - a tensor

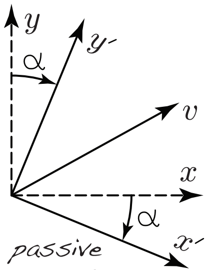
$$M(\alpha\vec{a}+\beta\vec{b}) = \alpha M(\vec{a}) + \beta M(\vec{b})$$

$$M\begin{pmatrix}x\\y\end{pmatrix} = \underbrace{M\begin{pmatrix}1\\0\end{pmatrix}}_{\vec{m}_1} x + \underbrace{M\begin{pmatrix}0\\1\end{pmatrix}}_{\vec{m}_2} y = \left(\begin{array}{c|c} m_{1x} & m_{2x} \\ m_{1y} & m_{2y}\end{array}\right)\begin{pmatrix}x\\y\end{pmatrix}$$

* Change of coordinates
- ~ two ways of thinking about transformations
  both yield the same transformed components
- ~ active: basis fixed, physically rotate vector
- ~ passive: vector fixed, physically rotate basis



active
transformation



passive
transformation

* Transformation matrix (active) - basis vs. components

$$(\vec{a}\,\vec{b}\,\vec{c}) = (\hat{x}\,\hat{y}\,\hat{z})\begin{pmatrix}a_x & b_x & c_x\\ a_y & b_y & c_y\\ a_z & b_z & c_z\end{pmatrix} \qquad \vec{x} = (\vec{a}\,\vec{b}\,\vec{c})\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = (\hat{x}\,\hat{y}\,\hat{z})\begin{pmatrix}x\\y\\z\end{pmatrix} \qquad \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}a_x & b_x & c_x\\ a_y & b_y & c_y\\ a_z & b_z & c_z\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$$

$$\vec{e}\,' = \vec{e}\,\mathcal{R} \qquad \vec{x} = \overbrace{\vec{e}\,'}\,\mathbb{X}' = \overbrace{\vec{e}\,\mathcal{R}}\underbrace{\mathcal{R}\,\mathbb{X}'} = \vec{e}\,\underbrace{\mathbb{X}} = \vec{x} \qquad \mathbb{X} = \mathcal{R}\;\;\mathbb{X}'$$

$$\boxed{\begin{array}{c}\vec{e}\,' = \vec{e}\,\mathcal{R} \\ \mathbb{X}' = \mathcal{R}^{-1}\mathbb{X}\end{array}}$$

* Orthogonal transformations
- ~ R is orthogonal if it 'preserves the metric' (has the same form before and after)

$$\vec{e}^{\,T}\cdot\vec{e} = \begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}\cdot(\hat{x}\;\hat{y}) = \begin{pmatrix}\hat{x}\cdot\hat{x} & \hat{x}\cdot\hat{y}\\ \hat{y}\cdot\hat{x} & \hat{y}\cdot\hat{y}\end{pmatrix} = \begin{pmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix} = g \qquad \vec{e}\,'^{T}\cdot\vec{e}\,' = \begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix}\cdot(\vec{a}\;\vec{b}) = \begin{pmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b}\end{pmatrix} = g'$$

$$\vec{e}\,' = \vec{e}\,\mathcal{R} \qquad \vec{e}\,'^{T}\cdot\vec{e}\,' = \mathcal{R}^T\,\vec{e}^{\,T}\cdot\,\vec{e}\,\mathcal{R} = \mathcal{R}^T g\,\mathcal{R} = g' \qquad g = g' \qquad \boxed{\mathcal{R}^T g\,\mathcal{R} = g}$$

- ~ equivlent definition in terms of components:

$$\vec{x}\cdot\vec{x} = \overbrace{\mathbb{X}^T}\,g\,\overbrace{\mathbb{X}} = \overbrace{\mathbb{X}'^T\mathcal{R}^T}\,\underbrace{\mathcal{R}^T g\,\mathcal{R}}\,\overbrace{\mathcal{R}\,\mathbb{X}'} = \mathbb{X}'^T\underbrace{g'}\,\mathbb{X}' \qquad \text{(metric invariant under rotations if } g=g')$$

- ~ starting with an orthonormal basis: $g = I \quad g_{ij} = \delta_{ij} \qquad \mathcal{R}^T\mathcal{R} = I \qquad \mathcal{R}^{-1} = \mathcal{R}^T$

* Symmetric / antisymmetric vs. Symmetric / orthogonal decomposition
- ~ recall complex numbers $u = \rho + i\phi \qquad \rho^* = \rho \qquad (i\phi)^* = -i\phi$

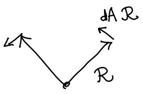
$$e^u = e^{\rho + i\phi} = r e^{i\phi} \qquad |e^{i\phi}|^2 = e^{-i\phi}e^{i\phi} = e^{i0} = 1$$

- ~ similar behaviour of symmetric / antisymmetric matrices

M artibrary matrix
T symmetric
A antisymmetric

S symmetric
R orthogonal

$$M = \begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a & (b+c)/2\\ (b+c)/2 & d\end{pmatrix} + \begin{pmatrix}0 & (b-c)/2\\ (c-b)/2 & 0\end{pmatrix} = T + A$$



$$e^M = 1 + M + \tfrac{1}{2!}M^2 + \tfrac{1}{3!}M^3 + \ldots \quad = e^{T+A} \neq e^T e^A$$

$$S = e^T = e^{VWV^{-1}} = V e^W V^{-1} \qquad R = e^A \qquad R^T R = (e^A)^T e^A = e^{A^T + A} = e^0 = I$$

$$\det\begin{pmatrix}e^{\lambda_1} & & \\ & e^{\lambda_2} & \\ & & \ddots\end{pmatrix} = e^{\lambda_1}\cdot e^{\lambda_2}\cdots = e^{\lambda_1+\lambda_2+\ldots} = e^{\operatorname{tr}\begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \ddots\end{pmatrix}} \qquad \det e^A = e^{\operatorname{tr} A} = e^0 = 1$$

Eigenparaphernalia

* illustration of symmetric matrix S with eigenvectors v, eigenvalues λ

$$S v = \lambda v$$

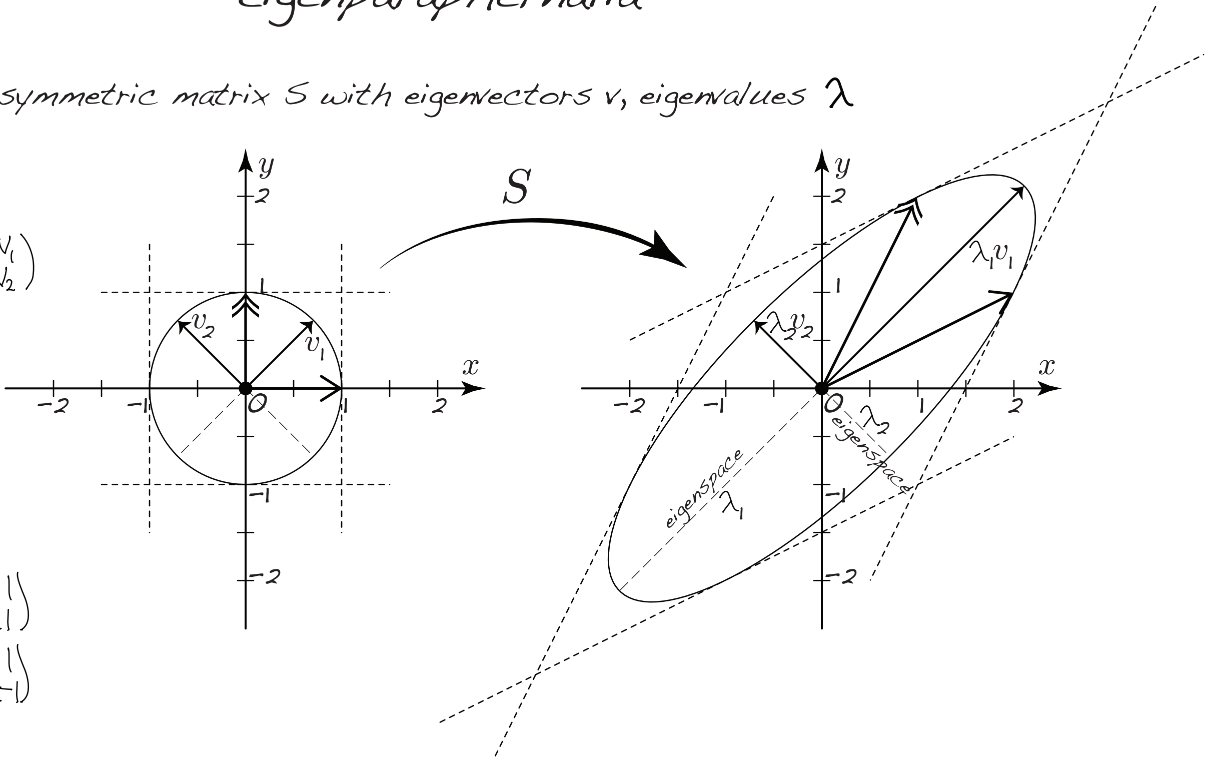
$$\begin{pmatrix}2&1\\1&2\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\lambda\begin{pmatrix}v_1\\v_2\end{pmatrix}$$

$$\begin{pmatrix}2&1\\1&2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$$

$$\begin{pmatrix}2&1\\1&2\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}$$

$$\begin{pmatrix}2&1\\1&2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}3\\3\end{pmatrix}=3\begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{pmatrix}2&1\\1&2\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}3\\3\end{pmatrix}=1\begin{pmatrix}1\\-1\end{pmatrix}$$



* similarity transform – change of basis (to diagonalize A)

$$S\,(v_1 v_2 \ldots) = (\vec{v}_1 \vec{v}_2 \ldots)\begin{pmatrix}\lambda_1 & \\ & \lambda_2 \ddots\end{pmatrix} \qquad S V = V W \qquad S = V W V^{-1} = V W V^T$$

* a symmetric matrix has real eigenvalues

$$S v = \lambda v \qquad v^{*T} S\, v = \lambda\, v^{*T} v$$

$$\lambda = \lambda^*$$

$$v^{*T} S = v^{*T} \lambda^* \qquad v^{*T} S\, v = \lambda^*\, v^{*T} v$$

~ what about a antisymmetric/orthogonal matrix?

* eigenvectors of a symmetric matrix with distinct eigenvalues are orthogonal

$$v^T S = (S^T v)^T = (S v)^T = (\lambda v)^T = v^T \lambda$$

$$\lambda_1\, v_1 \cdot v_2 = (v_1^T S)\, v_2 = v_1^T (S v_2) = v_1 \cdot v_2\, \lambda$$

$$v_1 \cdot v_2\, (\lambda_1 - \lambda_2) = 0 \qquad \text{if } \lambda_1 \neq \lambda_2 \text{ then } v_1 \cdot v_2 = 0.$$

* singular value decomposition (SVD)

~ transformation from one orthogonal basis to another

$$M = R\overbrace{S} = \overbrace{\underbrace{RV}WV^T} = \underbrace{U}W V^T$$

~ extremely useful in numerical routines

M	arbitrary matrix
R	orthogonal
S	symmetric
W	diagonal matrix
V	orthogonal (domain)
U	orthogonal (range)

Section 1.2 - Differential Calculus

* differential operator

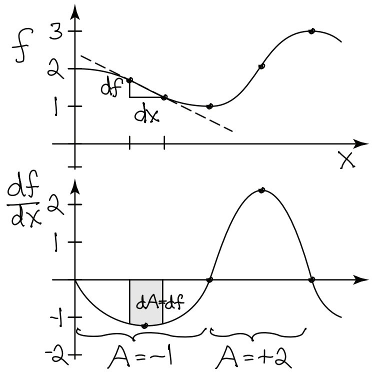
~ ex. $u = x^2 \quad du = dx^2 = 2x\,dx$ $\quad d \equiv \lim_{\Delta \to 0} \Delta \approx 0$

or $d(\sin x^2) = \cos(x^2)\,d\,x^2 = \cos x^2 \cdot 2x \cdot dx$

~ df and dx connected - refer to the same two endpoints

~ made finite by taking ratios (derivative or chain rule) or inifinite sum = integral (Fundamental Thereom of calculus)

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} \qquad \int_a^b \frac{df}{dx}\,dx = \int_a^b df = f\Big|_a^b$$



* scalar and vector fields - functions of position ($\vec{r}$)

~ "field of corn" has a corn stalk at each point in the field

~ scalar fields represented by level curves (2d) or surfaces (3d)

~ vector fields represented by arrows, field lines, or equipotentials

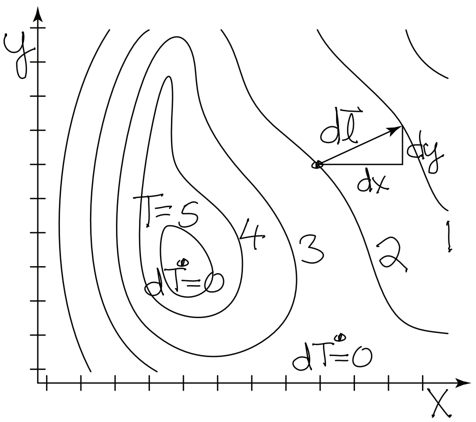
* partial derivative & chain rule

~ signifies one varying variable AND other fixed variables

~ notation determined by denominator; numerator along for the ride

~ total variation split into sum of variations in each direction

$$\frac{\partial u}{\partial x} \quad \left(\frac{\partial u}{\partial x}\right)_{y,z} \quad \partial_x u \quad u_{,x} \qquad \frac{\cdots}{\cdots} = \frac{dx}{\cdots}\frac{\cdots}{\partial x} + \frac{dy}{\cdots}\frac{\cdots}{\partial y} + \frac{dz}{\cdots}\frac{\cdots}{\partial z}$$



* vector differential - gradient

~ differential operator , del operator

$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz$$
$$= \underbrace{(\partial_x, \partial_y, \partial_z)}_{\nabla} T \cdot \underbrace{(dx, dy, dz)}_{d\vec{l}}$$

$$\boxed{\begin{aligned} d &= dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z} &&= d\vec{r}\cdot\nabla \\ \nabla &= \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} &&= \frac{d}{d\vec{r}} \\ d\vec{l} &= \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz &&= d\vec{r} \end{aligned}}$$

~ differential line element: $\cdot d\vec{l}$ and $\frac{}{d\vec{l}}$ transforms between $\hat{x},\hat{y},\hat{z} \leftrightarrow dx, dy, dz$ and $d \leftrightarrow \nabla$

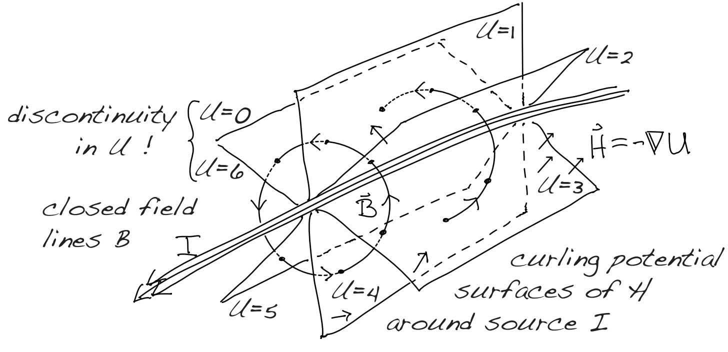
~ example: $d\,x^2y = 2xy\,dx + x^2dy = (2xy, x^2)\cdot(dx, dy)$

~ example: let $z = f(x,y)$ be the graph of a surface. What direction does ∇f point?

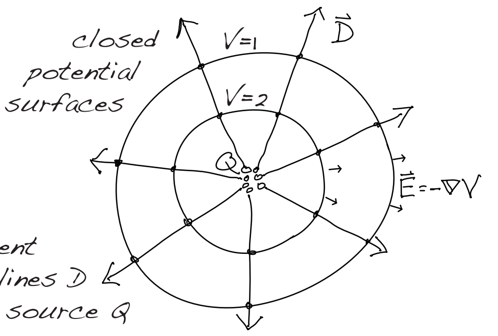
now let $g = z - f(x,y)$ so that $g = 0$ on the surface of the graph

then $\nabla g = (-f_{,x}, -f_{,y}, 1)$ is normal to the surface

* illustration of curl - flow sheets



* illustration of divergence - flux tubes



Higher Dimensional Derivatives

* curl - circular flow of a vector field

$$\nabla\times\vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ V_x & V_y & V_z \end{vmatrix} = \begin{matrix} \hat{x}\,(V_{z,y} - V_{y,z}) \\ +\hat{y}\,(V_{x,z} - V_{z,x}) \\ +\hat{z}\,(V_{y,x} - V_{x,y}) \end{matrix}$$

* divergence - radial flow of a vector field

$$\nabla\cdot\vec{V} = (\partial_x\,\partial_y\,\partial_z)\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = V_{x,x} + V_{y,y} + V_{z,z}$$

* product rules

~ how many are there?

~ examples of proofs

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}\,(\vec{a}\cdot\vec{b})$$

$$\vec{A}\times(\nabla\times\vec{B}) = \nabla\,(\vec{A}\cdot\overset{\downarrow}{\vec{B}}) - \overset{\downarrow}{\vec{B}}\,(\vec{A}\cdot\nabla)$$

$$\nabla\times(\overset{\downarrow}{\vec{A}}\times B) = \overset{\downarrow}{\vec{A}}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A})$$

$$\nabla(fg) = \nabla f\cdot g + f\cdot\nabla g$$

$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\leftrightarrow\vec{A})$$

$$\nabla\times(f\vec{A}) = \nabla f\times\vec{A} + f(\nabla\times\vec{A})$$

$$\nabla\times(\vec{A}\times\vec{B}) = (B\cdot\nabla)A - B(\nabla\cdot A) - (\vec{B}\leftrightarrow\vec{A})$$

$$\nabla\cdot(f\vec{A}) = \nabla f\cdot\vec{A} + f\,\nabla\cdot\vec{A}$$

$$\nabla\cdot(\vec{A}\times\vec{B}) = (\nabla\times\vec{A})\cdot\vec{B} - \vec{A}\cdot(\nabla\times\vec{B})$$

* second derivatives - there is really only ONE! (the Laplacian) $\boxed{\nabla^2 \equiv \nabla\cdot\nabla \equiv \partial_x^2 + \partial_y^2 + \partial_z^2}$

1) $\nabla\cdot(\nabla T) = \nabla^2 T$ ~ eg: $\nabla^2 T = 0$ no net curvature - stretched elastic band

$(\nabla\cdot\nabla)\vec{v} = \nabla^2\vec{v}$ ~ defined component-wise on v_x, v_y, v_z (only cartesian coords)

3), 5) $\nabla^2 = \nabla_{\parallel}^2 + \nabla_{\perp}^2$ ~ longitudinal / transverse projections $\nabla(\nabla\cdot\vec{v}) \equiv \nabla_{\parallel}^2\,\vec{v}$

$= \nabla(\nabla\cdot \;\; -\nabla\times\nabla\times$ $\quad\vec{K}\cdot\vec{K} = \vec{K}\vec{K}\cdot \; - \vec{K}\times(\vec{K}\times$ $\quad -\nabla\times\nabla\times\vec{v} \equiv -\nabla_{\perp}^2\vec{v}$

2), 4) $\nabla\times\nabla = 0$ ~ equality of mixed partials ($d^2=0$)

$$\nabla\times(\nabla T) = 0 \qquad \nabla\cdot(\nabla\times\vec{v}) = 0 \qquad \nabla\times\nabla = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ \partial_x & \partial_y & \partial_z \end{vmatrix} = \begin{matrix} +\hat{x}\,(\partial_y\partial_z - \partial_z\partial_y) \\ +\hat{y}\,(\partial_z\partial_x - \partial_x\partial_z) \\ +\hat{z}\,(\partial_x\partial_y - \partial_y\partial_x) \end{matrix}$$

* unified approach to all higher-order derivatives with differential operator

1) $d^2 = 0$ 2) $dx^2 = 0$ 3) $dx\,dy = -dy\,dx$ + differential (line, area, volume) elements

~ Gradient

$$df = f_{,x}\,dx + f_{,y}\,dy + f_{,z}\,dz = \nabla f\cdot d\vec{l} \qquad d\vec{l} = (dx, dy, dz) = d\vec{r}$$

~ Curl

$$\begin{aligned} d(\vec{A}\cdot d\vec{l}) &= d\,(A_x dx + A_y\,dy + A_z\,dz) \\ &= A_{x,x}\,\overset{0}{\cancel{dx\,dx}} + A_{x,y}\,dy\,dx + A_{x,z}\,dz\,dx \\ &\quad + A_{y,x}\,dx\,dy + A_{y,y}\,\overset{0}{\cancel{dy\,dy}} + A_{y,z}\,dz\,dy \\ &\quad + A_{z,x}\,dx\,dz + A_{z,y}\,dy\,dz + A_{z,z}\,\overset{0}{\cancel{dz\,dz}} \\ &= (A_{z,y} - A_{y,z})\,dy\,dz + (A_{x,z} - A_{z,x})\,dz\,dx + (A_{y,x} - A_{x,y})\,dx\,dy \end{aligned}$$

$$d(\vec{A}\cdot d\vec{l}) = (\nabla\times\vec{A})\cdot d\vec{a}$$

$$d\vec{a} = (dy\,dz, dz\,dx, dx\,dy) = \tfrac{1}{2}\,d\vec{l}\times d\vec{l} = d^2\vec{r}$$

~ Divergence

$$\begin{aligned} d(\vec{B}\cdot d\vec{a}) &= d\,(B_x\,dy\,dz + B_y\,dz\,dx + B_z\,dx\,dy) \\ &= B_{x,x}\,dx\,dy\,dz + B_{x,y}\,dy\,\overset{0}{\cancel{dy}}\,dz + B_{x,z}\,dz\,\overset{0}{\cancel{dy}}\,dz \\ &\quad + B_{y,x}\,dx\,\overset{0}{\cancel{dz}}\,dx + B_{y,y}\,dy\,dz\,dx + B_{y,z}\,dz\,\overset{0}{\cancel{dz}}\,dx \\ &\quad + B_{z,x}\,dx\,\overset{0}{\cancel{dx}}\,dy + B_{z,y}\,dy\,\overset{0}{\cancel{dx}}\,dy + B_{z,z}\,dz\,dx\,dy. \\ &= (B_{x,x} + B_{y,y} + B_{z,z})\,dx\,dy\,dz \end{aligned}$$

$$d(\vec{B}\cdot d\vec{a}) = \nabla\cdot\vec{B}\;d\tau \qquad d\tau = \tfrac{1}{6}\,d\vec{l}\cdot d\vec{l}\times d\vec{l} = d^3\vec{r}$$

$$\boxed{\nabla f = \frac{df}{d\vec{l}} = \frac{df}{d\vec{r}}} \qquad \boxed{\nabla\times\vec{A} = \frac{d(\vec{A}\cdot d\vec{l})}{d\vec{a}} = \frac{d\,(d\vec{r}\cdot\vec{A})}{d^2\vec{r}}} \qquad \boxed{\nabla\cdot\vec{B} = \frac{d(\vec{B}\cdot d\vec{a})}{d\tau} = \frac{d(d^2\vec{r}\cdot\vec{B})}{d^3\vec{r}}}$$

Section 1.3 - Integration

* different types of integration in vector calculus

1-dim: $\omega^{(1)} = \lambda dl\,,\ \varphi d\vec{l}\,,\ \vec{A} dl\,,\ \vec{A}\cdot d\vec{l}\,,\ \vec{A}\times d\vec{l}$

2-dim: $\omega^{(2)} = \sigma da\,,\ \sigma d\vec{a}\,;\ \vec{B} da\ \ \vec{B}\cdot d\vec{a}\,,\ \vec{B}\times d\vec{a}$

3-dim: $\omega^{(3)} = \ \rho d\tau\,,\quad \vec{F} d\tau$

Flow: $\mathcal{E}_A = \int \tilde{A} = \int \vec{A}\cdot d\vec{l}$

Flux: $\Phi_B = \int \tilde{B} = \int \vec{B}\cdot d\vec{a}$

Substance: $Q_\rho = \int \tilde{\rho} = \int \rho d\tau$

~ "differential forms" are everything after the '$\int$'
all have a 'd' somewhere inside

$d\vec{l}_{rec} = \hat{x} dx + \hat{y} dy + \hat{z} dz$

$d\vec{a}_{rec} = \hat{x} dy dz + \hat{y} dz dx + \hat{z} dx dy$

$d\tau_{rec} = dx\ dy\ dz$

~ often $d\vec{l}\,, d\vec{a}\,, d\tau$ are burried inside of another 'd'

current element $dq \equiv q_i^{(0)},\ \lambda dl^{(1)},\ \sigma da^{(2)},\ \rho d\tau^{(3)}$

charge element $d\vec{q} \equiv \vec{v} q_i,\ I d\vec{l}\,,\ \vec{K} da\,,\ \vec{J} d\tau$

~ two types of regions:

over the region R: $\int_R \omega$ (open region)

over the boundary ∂R of R: $\oint_{\partial R} \omega$ (closed region)

* recipe for ALL types of integration

coordinates on path/surface/volume

1-d $\mathcal{P}$: $\vec{r}(t)$

2-d $\mathcal{S}$ $\vec{r}(s,t)$

3-d $\mathcal{V}$ $\vec{r}(s,t,u)$

boundary of coordinates

$\int_{s=a}^{b} \int_{t=t(s)}^{t_1(s)}$

a) Parametrize the region
- ~ parametric vs relations equations of a region
- ~ boundaries translate to endpoints on integrals

b) Pull back the paramters
- ~ x,y,z become functions of s,t,u
- ~ differentials: dx,dy,dz become ds,dt,du
- ~ reduce using the chain rule

$d\vec{l} = \frac{d\vec{r}}{dt} dt$

$d\vec{a} = \frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t} ds\, dt$

$d\tau = \frac{\partial\vec{r}}{\partial s}\cdot\frac{\partial\vec{r}}{\partial t}\times\frac{\partial\vec{r}}{\partial u} ds\, dt\, du$

$x = x(t)$ $dx = x' dt$

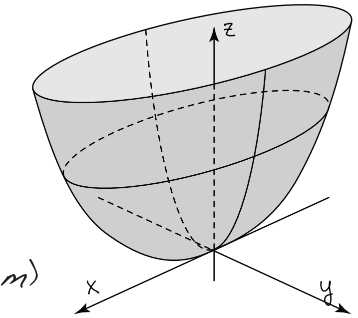
$y = y(t)$ $dy = y' dt$

$z = z(t)$ $dz = z' dt$

$$\int_R \vec{A}\cdot d\vec{l} = \int_{\vec{x}(t)} A_x(x,y,z)\,dx + A_y(x,y,z)\,dy + A_z(x,y,z)\,dz$$

$$= \int_{t=a}^{b} A_x(x(t),y(t),z(t))\frac{dx}{dt}dt + A_y(x(t),y(t),z(t))\frac{dy}{dt}dt$$

c) Integrate 1-d integrals using calculus of one variable



* example: line & surface integrals on a paraboloid (Stoke's theorem)

$\vec{A} = yz\hat{x}$

$0<z<1$

S: $z = \frac{1}{4}x^2 + y^2 = s^2(c_\phi^2 + s_\phi^2)$

∂S: $1 = \frac{1}{4}x^2 + y^2$

$x = 2sc_\phi$

$y = s\cdot s_\phi$

$z = s^2$

$dx = 2ds\,c_\phi + -2s\,s_\phi\, d\phi$

$dy = ds\, s_\phi + s\, c_\phi\, d\phi$

$dz = 2s\cdot ds$

$$d\vec{l} = \underbrace{\frac{\partial\vec{r}}{\partial s} ds}\ +\ \underbrace{\frac{\partial\vec{r}}{\partial\phi} d\phi}_{\text{'partials'}} = d\vec{l}_s + d\vec{l}_\phi$$

$$d\vec{a} = d\vec{l}_s\times d\vec{l}_\phi = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 2c_\phi & s_\phi & 2s \\ -2ss_\phi & sc_\phi & 0 \end{vmatrix} ds\, d\phi$$

$$= \left(-\hat{x}\,2s^2c_\phi - \hat{y}\,4s^2s_\phi + \hat{z}\,2s\right) ds\, d\phi$$

∂S: $\vec{r}(s,\phi)$ $s=1$ $ds=0$ $d\vec{l} = d\vec{l}_\phi\ (s=1)$

$$\oint_{\partial S}\vec{A}\cdot d\vec{l} = \int_{\partial S} yz\,dx = -2\int_0^{2\pi} \overset{1/2}{\widetilde{s_\phi^2}}\, d\phi = -2\pi$$

$$\int_S \nabla\times\vec{A}\cdot d\vec{a} = \int_S (\hat{y}\partial_z - \hat{z}\partial_y)yz\cdot d\vec{a} = \int_S y\,da_y - z\,da_z$$

$$= \int_0^1\int_0^{2\pi} (s\cdot s_\phi\cdot -4s^2s_\phi - s^2\cdot 2s)\,ds\,d\phi$$

$$= \int_0^1 ds \int_0^{2\pi} -4s^3\underbrace{s_\phi^2}_{1/2} - 2s^3\ d\phi$$

$$= \int_0^1 -4s^3\cdot ds\cdot 2\pi = \frac{-4s^4}{4}\Big|_0^1\cdot 2\pi = -2\pi$$

* alternate method: substitute for dx, dy, dz (antisymmetric)

$$\int_S y\,dz\,dx - z\,dx\,dy = \int_S s s_\phi\cdot 2s\,ds\cdot(2c_\phi ds - 2ss_\phi d\phi)$$
$$- s^2(2c_\phi ds - 2ss_\phi d\phi)(s_\phi ds + sc_\phi d\phi)$$

$$= \int_S -4s^3s_\phi^2\,ds\,d\phi - 2s^3c_\phi^2\,ds\,d\phi + 2s^3s_\phi^2\,\underbrace{d\phi\,ds}_{-ds\,d\phi}$$

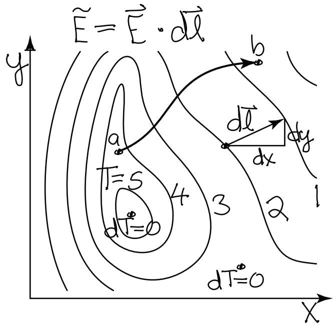
$$= \int_S (-6s_\phi^2 - 2c_\phi^2)\,s^3\,ds\,d\phi$$

Flux, Flow, and Substance

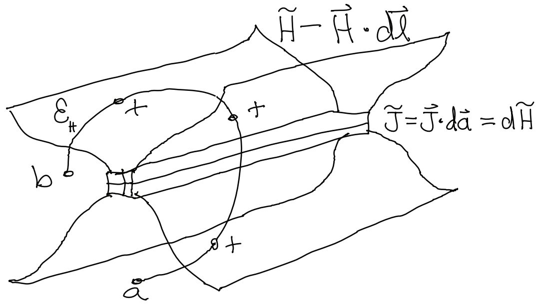
* Differential forms		Name	Geometrical picture
scalar:	$\varphi = \varphi(x)$		level curves
vector:	$\text{đ}\mathcal{E} = \vec{A}\cdot d\vec{l} = A_x\,dx + A_y\,dy + A_z\,dz$		equipotentials (flow sheets)
pseudovector:	$\text{đ}\Phi = \vec{B}\cdot d\vec{a} = B_x\,dy\,dz + B_y\,dz\,dx + B_z\,dx\,dy$		fieldlines (flux tubes)
pseudoscalar:	$\text{đ}q = \rho\,d\tau = \rho\,dx\,dy\,dz$		boxes of substance

* Derivative 'd'			
scalar:	$d\,\varphi = \nabla\varphi\cdot d\vec{l}$	grad	same equipotential surfaces
vector:	$d\,\vec{A}\cdot d\vec{l} = \nabla\times\vec{A}\cdot d\vec{a}$	curl	flux tubes at end of sheets
pseudovector:	$d\,\vec{B}\cdot d\vec{a} = \nabla\cdot\vec{B}\,d\tau$	div	boxes at the end of flux tubes
pseudoscalar:	$d\,\rho\,d\tau = 0$		

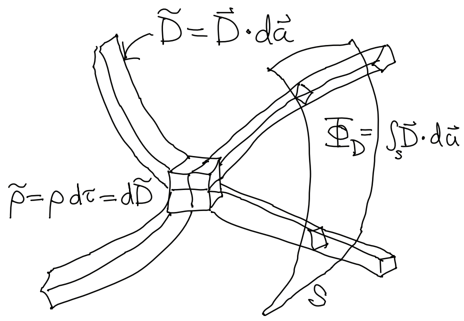
* Definite integral			
scalar:			
vector:	$\mathcal{E} = \int_P \text{đ}\mathcal{E} = \int_P \vec{A}\cdot d\vec{l}$	flow	# of surfaces pierced by path
pseudovector:	$\Phi = \int_S \text{đ}\Phi = \int_S \vec{B}\cdot d\vec{a}$	flux	# of tubes piercing surface
pseudoscalar:	$Q = \int_V \text{đ}q = \int_V \rho\,d\tau$	subst	# of boxes inside volume



$$\Delta f = \int_a^b df = f\Big|_a^b = -4$$
$$\oint df = \Delta f = 0$$
$$df = \nabla f\cdot d\vec{l}$$



$$\mathcal{E}_H = \int_a^b \tilde{H} = \int_a^b \vec{H}\cdot d\vec{l} = +3$$
$$\mathcal{E}_H = \oint_{\partial R} \tilde{H} = \int_R d\tilde{H} = \int \tilde{J} = I = +4$$
$$d\tilde{H} = d(\vec{H}\cdot d\vec{l}) = (\nabla\times\vec{H})\cdot d\vec{a} = \vec{J}\cdot d\vec{a} = \tilde{J}$$



$$\Phi_D = \int_S \vec{D}\cdot d\vec{a} = \int_S \tilde{D} = +2.$$
$$\Phi_D = \oint_{\partial R} \tilde{D} = \int_R d\tilde{D} = \int_R \tilde{\rho} = Q = +4$$
$$d\tilde{D} = d(\vec{D}\cdot d\vec{a}) = \nabla\cdot\vec{D}\,d\tau = \rho\,d\tau = \tilde{\rho}$$

* Stoke's theorem

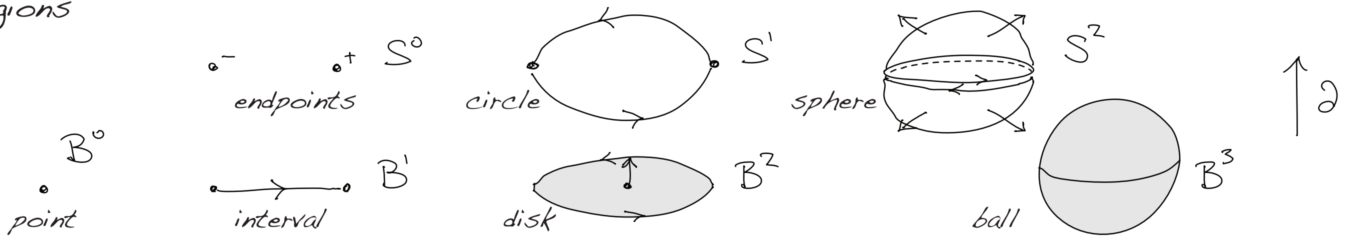
\# of flux tubes puncturing disk (S) bounded by closed path
EQUALS # of surfaces pierced by closed path (∂S)
~ each surface ends at its SOURCE flux tube

* Divergence theorem

\# of substance boxes found in volume (R) bounded by closed surface
EQUALS # of flux tubes piercin the closed surface (∂R)
~ each flux tube ends at its SOURCE substance box

Section 1.3.2-5 - Region | Form = Integral

* Regions



~ definition of boundary operator '∂'
'closed' region (cycle): $\partial S = 0$

~ a boundary is always closed $\boxed{\partial\partial R = 0}$

~ is every closed closed region a boundary?
$\partial S = 0 \iff S = \partial R$

~ a room (walls, window, ceiling, floor)
is CLOSED if all doors, windows closed
is OPEN if the door or window is open;

~ what is the boundary?

~ think of a surface that has loops
that do NOT wrap around disks!

* Forms - see last notes

~ combinations of scalar/vector fields and differentials so they can be integrated

~ pictoral representation enables 'integration by eye'

RANK	NOTATION		REGION	VISUAL REP.		DERIVATIVE
scalar	$\omega^{(0)} = f$	∂	Q point	level surfaces	d	$d\omega^{(0)} = \nabla f \cdot d\vec{l}$
vector	$\omega^{(1)} = \tilde{A} = \vec{A} \cdot d\vec{l}$	∂	P path	flow sheets	d	$d\omega^{(1)} = \nabla \times \vec{A} \cdot d\vec{a}$
p-vector	$\omega^{(2)} = \tilde{B} = \vec{B} \cdot d\vec{a}$	∂	S surface	flux tubes	d	$d\omega^{(2)} = \nabla \cdot B \, d\tau$
p-scalar	$\omega^{(3)} = \tilde{\rho} = \rho d\tau$	∂	V volume	subst boxes	d	$d\omega^{(3)} = 0$
					edge of the world!	

~ properties of differential operator 'd'
transforms form into higher-dimensional form, sitting on the boundary

~ Poincare lemma $\boxed{dd\omega = 0}$ $\nabla \times \nabla V = 0$ $\nabla \cdot \nabla \times \vec{A} = 0$

~ converse - existance of potentials $V, \vec{A}$

$$d\omega = 0 \iff \omega = d\alpha \qquad \nabla \times E = 0 \iff E = -\nabla V \qquad \nabla \cdot \vec{B} = 0 \iff \vec{B} = \nabla \times \vec{A}$$

for space without any n-dim 'holes' in it

* Integrals - the overlap of a region on a form = integral of form over region

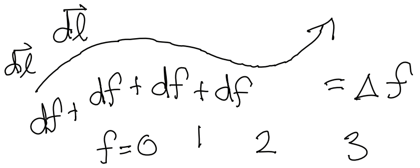
~ regions and forms are dual - they combine to form a scalar

~ generalized Stoke's therem:
'∂' and 'd' are adjoint operators - they have the same effect in the integral

$$\int_R d\omega = \oint_{\partial R} \omega \qquad \text{note: } 0 = \int_{\partial\partial R} \omega = \int_{\partial R} d\omega = \int_R dd\omega = 0$$

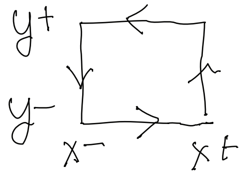
Generalized Stokes Theorem

* Fundamental Theorem of Vector Calculus: 0d–1d



$$\int_a^b \nabla\varphi\cdot d\vec{l} = \int_a^b df = f(b) - f(a)$$

* Stokes' Thereom: 1d–2d

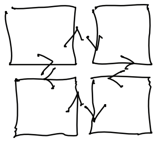


$$\nabla\times\vec{A}\cdot d\vec{a} = \frac{\partial A_y}{\partial x}dx\,dy - \frac{\partial A_x}{\partial y}dx\,dy + \ldots$$

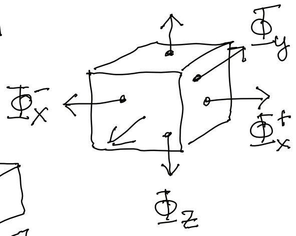
$$= A_y(x^+)dy + A_y(x^-)(-dy) + A_x(y^+)(-dx) + A_x(y^-)dx + \ldots$$

$$= \Sigma\,\vec{A}\cdot d\vec{l} \text{ around boundary}$$

+ other faces



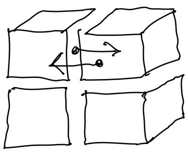
* Gaus' Thereom: 2d–3d (divergence theorem)



$$\nabla\cdot\vec{B}\,d\tau = \frac{\partial B_x}{\partial x}dx\,dy\,dz + \frac{\partial B_y}{\partial y}dy\,dz\,dx + \frac{\partial B_z}{\partial z}dz\,dx\,dy$$

$$= B_x(x^+)dy\,dz + B_x(x^-)(-dy\,dz) + 4 \text{ other faces}$$

$$= \Sigma\,\vec{B}\cdot d\vec{a} \text{ around boundary}$$



* note: all interior $f(x)$, flow, and flux cancel at opposite edges
* proof of converse Poincare lemma: integrate form out to boundary
* proof of gen. Stokes theorem: integrate derivative out to the boundary

$$\int_R d\omega = \oint_{\partial R}\omega \iff \int_P \nabla\varphi\cdot d\vec{l} = \oint_{\partial P}\varphi \qquad \int_S \nabla\times\vec{A}\cdot d\vec{a} = \oint_{\partial S}\vec{A}\cdot d\vec{l} \qquad \int_R \nabla\cdot\vec{B}\,d\tau = \oint_{\partial R}\vec{B}\cdot d\vec{a}$$

* example – integration by parts

$$\nabla\cdot\left(\frac{\hat{r}}{r^2}f\right) = \left(\nabla\cdot\frac{\hat{r}}{r^2}\right)f + \frac{\hat{r}}{r^2}\cdot\nabla f$$

$$\int_v \frac{\hat{r}}{r^2}\cdot\nabla f\,d\tau = \int_v \nabla\cdot\left(\frac{\hat{r}}{r^2}f\right)\cdot d\tau - \int_v\left(\nabla\cdot\frac{\hat{r}}{r^2}\right)f\,d\tau$$

$$\int_v \frac{1}{r^2}\frac{\partial f}{\partial r}r^2 dr\cdot d\Omega = \oint_{\partial v} d\vec{a}\cdot\frac{\hat{r}}{r^2}f - \int_v 4\pi\,\delta^3(\vec{r})\,f\,d\tau$$

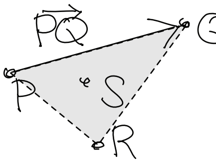
$$\int d\Omega\int_{r=0}^{R} df = \int r^2 d\Omega\,\hat{r}\cdot\frac{\hat{r}}{r^2}f - 4\pi f(0)$$

$$\int d\Omega\; f(R) - f(0) = \int d\Omega\; f(R,\theta,\phi) - 4\pi f(0)$$

$$4\pi\left[\langle f\rangle_R - f(0)\right] = 4\pi\left[\langle f\rangle_R - f(0)\right]$$

Section 1.4 - Affine Spaces

* Affine Space - linear space of points



$$S = \alpha P + \beta Q + \gamma R$$
$$\alpha + \beta + \gamma = 1$$

POINTS vs VECTORS

~ operations $Q - P = \vec{V}$ $\qquad \vec{W} = \alpha\vec{u} + \beta\vec{v}$

$P + \vec{V} = Q$

~ points are invariant under translation of the origin, but coordinates depend on the origin

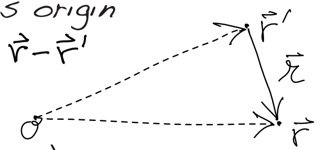
~ a point may be specified by its 'position vector' (arrow from the origin to the point)

cumbersome picture: many meaningless arrows from a meaningless origin

position vector: field point $\vec{r} = (x, y, z)$ $\quad$ source pt $\vec{r}\,' = (x', y', z')$

displacement vector: $\vec{\mathscr{r}} \equiv \vec{r} - \vec{r}\,'$

differential: $d\vec{l} = \frac{\partial \vec{r}}{\partial q} dq = \vec{b}\, dq$



~ the only operation on points is a weighted average (affine combination)

weight $w=0$ forms a vector and $w=1$ forms a point

~ transformation: affine vs linear

~ basis (independent): $N+1$ vs N

~ decomposition: coordinates vs components

- they appear the same for cartesian systems!
- coordinates are scalar fields $q^i(\vec{r})$
- they parametrize space

$$\begin{pmatrix} R & \vec{\epsilon} \\ 000 & 1 \end{pmatrix}\begin{pmatrix} \vec{r} \\ 1 \end{pmatrix} = \begin{pmatrix} R\vec{r} + \vec{\epsilon} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} R & \vec{\epsilon} \\ 000 & 1 \end{pmatrix}\begin{pmatrix} \vec{v} \\ 0 \end{pmatrix} = \begin{pmatrix} R\vec{v} \\ 0 \end{pmatrix}$$

* Rectangular, Cylindrical and Spherical coordinate transformations

~ math: 2-d -> N-d $\qquad$ physics: 3d + azimuthal symmetry

~ singularities on z-axis and origin

$S_\theta \equiv \sin\theta$
$C_\theta \equiv \cos\theta$

$$(\hat{s}, \hat{\phi}, \hat{z}) = (\hat{x}, \hat{y}, \hat{z}) \overbrace{\begin{pmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{R_{\hat{z}}(\phi)}$$

$x = s \cdot C_\phi$
$y = s \cdot S_\phi$

$s = r \cdot S_\theta$
$z = r \cdot C_\theta$

rect.		cyl.		sph.
x	$=$	$s \cdot C_\phi$	$=$	$r \cdot S_\theta \cdot C_\phi$
y	$=$	$s \cdot S_\phi$	$=$	$r \cdot S_\theta \cdot S_\phi$
z	$=$	z	$=$	$r \cdot C_\theta$

$$(\hat{r}\,\hat{\theta}\,\hat{\phi}) = (\hat{s}\,\hat{\phi}\,\hat{z}) \overbrace{\begin{pmatrix} S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \\ C_\theta & -S_\theta & 0 \end{pmatrix}}^{R_{\hat{\phi}}(\theta)} = (\hat{x}\,\hat{y}\,\hat{z}) \overbrace{R_{\hat{z}}(\phi) \cdot R_{\hat{\phi}}(\theta)}^{R(\theta,\phi)}$$

$$\| $$

$$\begin{pmatrix} S_\theta C_\phi & C_\theta C_\phi & -S_\phi \\ S_\theta S_\phi & C_\theta S_\phi & C_\phi \\ C_\theta & -S_\theta & 0 \end{pmatrix}$$

$d\vec{l}_{rec} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$

$d\vec{l}_{cyl} = \hat{s}\,ds + \hat{\phi}\,s\,d\phi + z\,dz$

$d\vec{l}_{sph} = \hat{r}\underbrace{dr}_{dl_r} + \hat{\theta}\underbrace{r\,d\theta}_{dl_\theta} + \hat{\phi}\underbrace{r\sin\theta\,d\phi}_{dl_\phi}$

$d\vec{a}_{rec} = \hat{x}\,dy\,dz + \hat{y}\,dz\,dx + \hat{z}\,dx\,dy$

$d\vec{a}_{cyl} = \hat{s}\,s\,d\phi\,dz + \hat{\phi}\,dz\,ds + \hat{z}\,ds\,s\,d\phi$

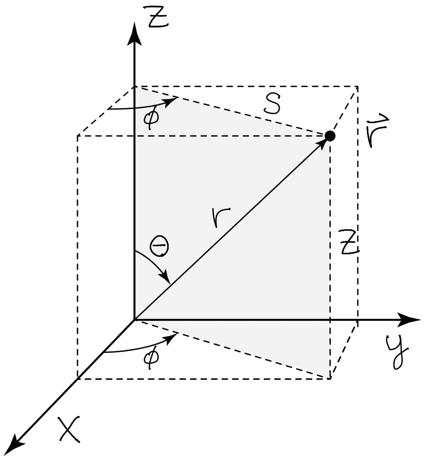
$d\vec{a}_{sph} = \hat{r}\,r\,d\theta\,r\sin\theta\,d\phi + \hat{\theta}\,r\sin\theta\,d\phi\,dr + \hat{\phi}\,dr\,r\,d\theta$

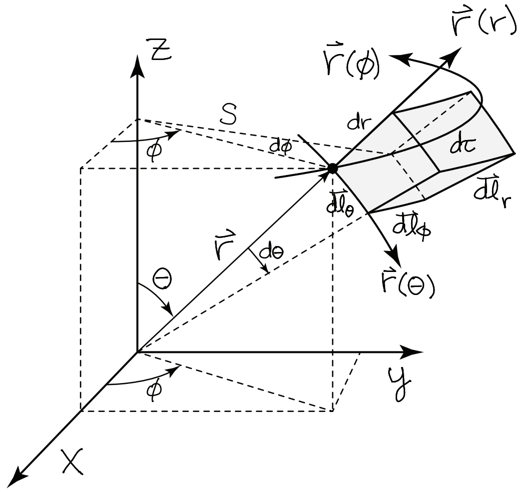
$d\tau_{rec} = dx\,dy\,dz$

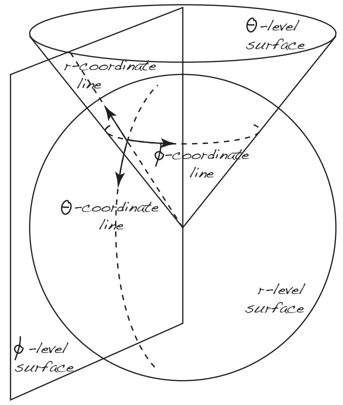
$d\tau_{cyl} = ds \cdot s\,d\phi \cdot dz$

$d\tau_{sph} = dr \cdot r\,d\theta \cdot r\sin\theta\,d\phi$

$= r^2 dr\,d\Omega$







Curvilinear Coordinates

* coordinate surfaces and lines
- ~ each coordinate is a scalar field $q(\vec{r})$
- ~ coordinate surfaces: constant q^i
- ~ coordinate lines: constant q^j, q^k

* coordinate basis vectors

$q^i \sim \{u, v, w\}$ ~ generalized coordinates

$\vec{b}_i = \left(\frac{\partial \vec{r}}{\partial q^i}\right)_{q^j, q^k} \sim \{\hat{u}f, \hat{v}g, \hat{w}h\}$ ~ contravariant basis

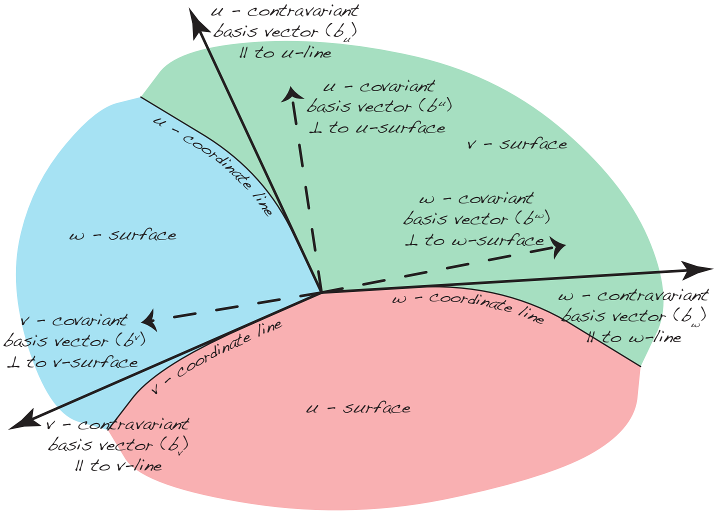
$\vec{b}^i = \nabla q^i \sim \{\hat{u}/f, \hat{v}/g, \hat{w}/h\}$ ~ covariant basis

$h_i = |\vec{b}_i| \sim \{f, g, h\}$ ~ scale factor

$\hat{e}_i = \vec{b}_i / h_i \sim \{\hat{u}, \hat{v}, \hat{w}\}$ ~ unit vector

$g_{ij} = \vec{b}_i \cdot \vec{b}_j \sim \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix}$ ~ metric (dot product)

$\vec{\Gamma}_{ij} = \frac{\partial \vec{b}_j}{\partial q^i} = \vec{b}_k \Gamma^k_{ij}$ ~ Christoffel symbols - derivative of basis vectors



* differential elements (orthogonal coords)

$$d\vec{l} = \frac{\partial \vec{r}}{\partial q^1} dq^1 + \frac{\partial \vec{r}}{\partial q^2} dq^2 + \frac{\partial \vec{r}}{\partial q^3} dq^3 = \vec{b}_i dq^i$$

$$= \hat{e}_1 \underbrace{h_1 dq^1}_{dl_1} + \hat{e}_2 \underbrace{h_2 dq^2}_{dl_2} + \hat{e}_3 \underbrace{h_3 dq^3}_{dl_3}$$

$$d\vec{a} = \tfrac{1}{2} d\vec{l} \times d\vec{l} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ h_1 dq^1 & h_2 dq^2 & h_3 dq^3 \\ h_1 dq^1 & h_2 dq^2 & h_3 dq^3 \end{vmatrix}$$

$$= \hat{e}_1 \cdot h_2 dq^2 \cdot h_3 dq^3 + \hat{e}_2 \cdot h_3 dq^3 \cdot h_1 dq^1 + \hat{e}_3 \cdot h_1 dq^1 \cdot h_2 dq^2$$

$$d\tau = \tfrac{1}{2} d\vec{l} \cdot d\vec{a} = \tfrac{1}{2} d\vec{l} \cdot d\vec{l} \times d\vec{l} = h_1 dq^1 \cdot h_2 dq^2 \cdot h_3 dq^3$$

* example $(c_\phi \equiv \cos\phi)$

$x = s\, c_\phi \qquad dx = c_\phi\, ds - s\, s_\phi\, d\phi$

$y = s\, s_\phi \qquad dy = s_\phi\, ds + s\, c_\phi\, d\phi$

$$d\vec{l} = \hat{x}\,dx + \hat{y}\,dy = (\hat{x} c_\phi + \hat{y} s_\phi) ds + (\hat{x} s_\phi - \hat{y} c_\phi) s\, d\phi$$

$$= \hat{s}\,ds + \hat{\phi}\, s\, d\phi \qquad (\hat{s}\ \hat{\phi}) = (\hat{x}\ \hat{y}) \begin{pmatrix} c_\phi & -s_\phi \\ s_\phi & c_\phi \end{pmatrix}$$

$s^2 = x^2 + y^2 \qquad 2s\,ds = 2x\,dx + 2y\,dy$

$y = x \tan\phi \qquad dy = dx \tan\phi + x \sec^2\phi\, d\phi$

$$d\phi = \frac{-y}{s^2} dx + \frac{x}{s^2} dy$$

$$\nabla s = \frac{x}{s}\hat{x} + \frac{y}{s}\hat{y} = c_\phi \hat{x} + s_\phi \hat{y} = \hat{s}$$

$$\nabla \phi = \frac{-y}{s^2}\hat{x} + \frac{x}{s^2}\hat{y} = \frac{-s_\phi \hat{x} + c_\phi \hat{y}}{s} = \frac{\hat{\phi}}{s}$$

* formulas for vector derivatives in orthogonal curvilinear coordinates

$$df = \frac{\partial f}{\partial q^i} dq^i = \frac{\partial f}{h_i \partial q^i} \cdot h_i dq^i = \nabla f \cdot d\vec{l} \qquad \nabla f = \frac{df}{d\vec{r}} = \frac{\hat{e}_i}{h_i} \frac{\partial}{\partial q^i} f$$

$$d(\vec{A} \cdot d\vec{l}) = d(A_k h_k dq^k) = \frac{\partial}{\partial q^j}(h_k A_k)\, dq^j dq^k$$

$$= \varepsilon_{ijk} \frac{\partial (h_k A_k)}{h_j h_k \partial q^k} d\vec{a}_i = (\nabla \times \vec{A}) \cdot d\vec{a}$$

$$\nabla \times \vec{A} = \frac{d(d\vec{r} \cdot \vec{A})}{d^2\vec{r}} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \partial/\partial q^1 & \partial/\partial q^2 & \partial/\partial q^3 \\ h_1 A^1 & h_2 A^2 & h_3 A^3 \end{vmatrix}$$

$$d(\vec{B} \cdot d\vec{a}) = d(B_i h_j dq^j h_k dq^k) = \frac{\partial}{\partial q^i}(h_j h_k B_i)\, dq^i dq^j dq^k$$

$$= \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial q} \frac{\partial (h_j h_k B_i)}{\partial q^i} d\tau = \nabla \cdot \vec{B}\, d\tau$$

$$\nabla \cdot \vec{B} = \frac{d(d^2\vec{r} \cdot \vec{B})}{d^3\vec{r}} = \frac{1}{h_1 h_2 h_3} \sum_i \frac{\partial}{\partial q^i}(h_j h_k B_i)$$

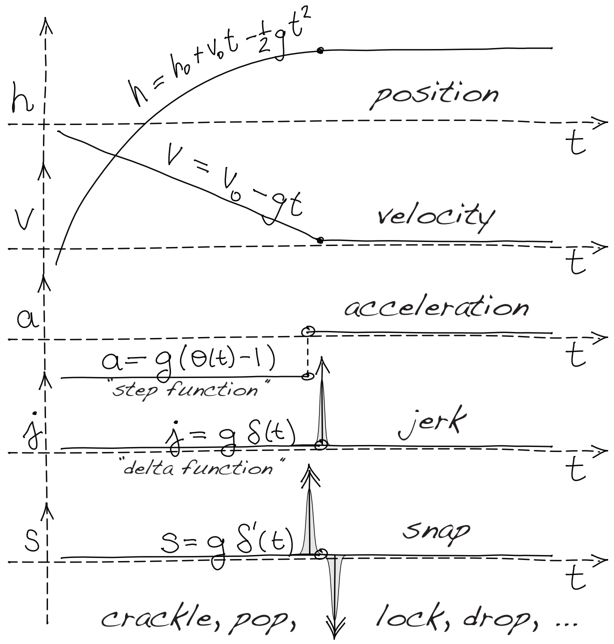
i,j,k cyclic

this formula does not work for $\nabla^2 \vec{B} \longrightarrow \nabla^2 f = \frac{1}{h_1 h_2 h_3} \sum_i \frac{\partial}{\partial q_i} \frac{h_j h_k}{h_i} \frac{\partial}{\partial q_i} f$

instead, use: $\nabla^2 = \nabla\nabla\cdot - \nabla\times\nabla\times$

Section 1.5 - Dirac Delta Distribution

* Newton's law: yank = mass × jerk
http://wikipedia.org/wiki/position_(vector)



* definition: $d\theta = \delta(x-x')dx$ is defined by its integral (a distribution, differential, or functional)

$$\int_a^b \underbrace{\delta(x)\,dx}_{d\theta \text{ "differential"}} = \int_a^b d\theta = \Theta(x)\Big|_a^b = \begin{cases} 1 & a<0<b \\ 0 & \text{otherwise} \end{cases}$$

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$ it is a "distribution," NOT a function!

* important integrals related to $\delta(x)$

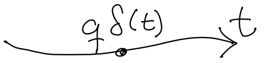
$$\int_{-\infty}^{\infty} \Theta(x)\, f(x)\,dx = \int_0^{\infty} f(x)\,dx$$ "mask"

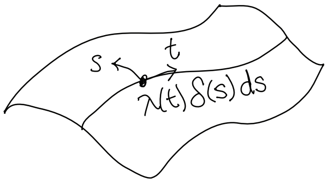
$$\int_{-\infty}^{\infty} \delta(x)\, f(x)\,dx = f(0)$$ "slit"

$$\int_{-\infty}^{\infty} \delta'(x) f(x)\,dx = f(x)\delta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\,\delta(x)\,dx = -f'(0)$$

* $\delta(x-x')$ is the an "undistribution" - it integrates to a lower dimension

$$\int_C dq = \int_C \lambda\, dl = \int_C q \underbrace{\delta(t)\,dt}_{d\theta} = q$$

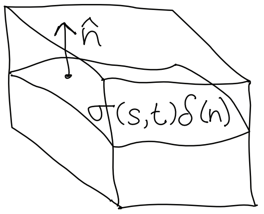




$$\int_A dq = \int_A \sigma\, da = \int_A \lambda(t) \underbrace{\delta(s)\,ds}_{d\theta}\, dt = \int_C \lambda(t)\,dt = q$$

$$\int_V dq = \int_V \rho\, d\tau = \int_V \sigma(s,t) \underbrace{\delta(n)\,dn}\, ds\, dt = \int_A \sigma\, da = q$$

$$\text{or} = \int_V q\, \delta^3(\vec{r}) = q \qquad \text{or} = \int_V \lambda\, \delta^2(\vec{r}) = q$$



* $\delta(x-x')$ gives rise to boundary conditions - integrate the diff. eq. across the boundary

$$\nabla \cdot \vec{D} = \rho = \sigma(s,t)\,\delta(n)$$

$$\nabla \to \hat{n} \cdot \Delta \quad \rho \to \sigma \quad \vec{J} \to \vec{K}$$

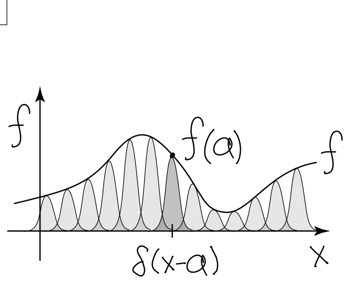
$$\int_{n=0^-}^{0^+} dn \left(\frac{\partial D_n}{\partial n} + \cancelto{0}{\frac{\partial D_s}{\partial s}} + \cancelto{0}{\frac{\partial D_t}{\partial t}} \right) = \int_{0^-}^{0^+} \sigma(s,t)\,\delta(n)\,dn$$

$$\boxed{\hat{n} \cdot \Delta\vec{D} = \sigma}$$

* $\delta(x-x')$ is the "kernel" of the identity transformation

$$f = I f \qquad f(x) = \underbrace{\int_{-\infty}^{\infty} dx'\, \delta(x-x')}_{\text{identity operator}} f(x')$$

(component form)



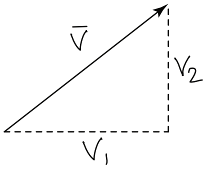
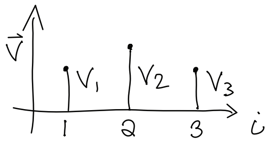
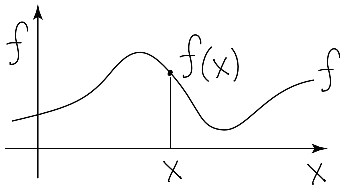
* $\delta(x-x')$ is the continuous version of the "Kroneker delta" δ_{ij}

$$a = I a \qquad a_i = \sum_{j=1}^{n} \delta_{ij}\, a_j \qquad \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

Linear Function Spaces

* functions as vectors (Hilbert space)

~ functions under pointwise addition have the same linearity property as vectors

	VECTORS	FUNCTIONS
~ addition	$\vec W = \vec V + \vec U \qquad W_i = V_i + U_i$	$h = f + g \qquad h(x) = f(x) + g(x)$
~ expansion	$\vec V = \underbrace{\sum_i}_{\text{index}} V_i = \underbrace{V_1}_{\text{component}} \underbrace{\hat e_1}_{\text{basis vector}} + V_2 \hat e_2 + \dots$	$f(x) = \underbrace{\int_{x'=-\infty}^{\infty}}_{\text{index}} \underbrace{f(x')}_{\text{component}} \cdot \underbrace{\delta(x-x')}_{\text{basis function}}$ or $f(x) = \sum_{i=0}^{\infty} \overbrace{f_i} \cdot \overbrace{\phi_i(x)}$
~ graph	 	
~ inner product (metric, symmetric bilinear product)	$\vec V \cdot \vec U = \sum_{i=1}^{n} V_i U_i$	$\langle f \| g \rangle = \int_{-\infty}^{\infty} dx\, f(x)\, g(x)$
~ orthonormality (independence)	$\hat e_i \cdot \hat e_j = \delta_{ij}$	$\int_{-\infty}^{\infty} \phi_i(x)\, \phi_j(x) = \delta_{ij} \qquad \int_{x'=-\infty}^{\infty} \delta(x-x')\, \delta(x'-y) = \delta(x-y)$
~ closure (completeness)	$\sum_{i=1}^{n} \hat e_i \hat e_i \cdot = I$	$\sum_{i=0}^{\infty} \phi_i(x)\, \phi_i(y) = \int_{x'=-\infty}^{\infty} \delta(x-x')\, \delta(x'-y) = \delta(x-y)$
~ linear operator (matrix)	$\vec u = A \vec v \qquad u_i = A_{ij} v_j$	$f = Hg \qquad f(x) = \int_{-\infty}^{\infty} dx'\, H(x,x')\, g(x')$
~ orthogonal rotation (change of coordinates) (Fourier transform)	$x' = Rx$ $R^T R = I$	$\tilde f(k) = \frac{1}{2\pi} \int dx\, e^{ikx} f(x)$ $\int dk\, e^{-ikx} e^{ikx'} = \int dk\, e^{-ik(x-x')} = 2\pi\, \delta(x-x')$
~ eigen-expansion (stretches) (principle axes)	$A \vec v = \vec v \lambda$ $A V = V W$	$H \phi(x) = \lambda \phi(x)$ (Sturm-Liouville problems)
~ gradient, functional derivative	$\nabla f = \frac{df}{d\vec r}$	$\frac{\delta F[\rho(x)]}{\delta \rho}$ (functional minimization)

* Sturm-Liouville equation - eigenvalues of function operators (2^{nd} derivative)

$$\mathcal{L}[y] = -\frac{d}{dx}\left[p(x) \frac{d}{dx} y\right] + q(x) = \lambda\, w(x)\, y \qquad \text{BC: } y(a),\ y(b)$$

~ there exists a series of eigenfunctions $y_n(x)$ with eigenvalues λ_n

~ eigenfunctions belonging to distinct eigenvalues are orthogonal $\langle y_i | y_j \rangle = \delta_{ij}$

Green Functions $G(x,x')$

- Green's functions are used to "invert" a differential operator
  - ~ they solve a differential equation by turning it into an integral equation

- You already saw them last year! (in Phy 232)
  - ~ the electric potential of a point charge

$$§1.5\!: \quad \nabla\cdot\frac{\hat r}{r^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = 0$$

a) $\frac{1}{r^2} \to \infty$ at $r=0$ "singularity"

b) $$\int_V \nabla\cdot\frac{\hat r}{r^2}\,d\tau = \oint_{\partial V} d\vec a\cdot\frac{\hat r}{r^2} = \oint_\Omega d\Omega\, r^2\frac{1}{r^2} = 4\pi$$

independent of volume if 0 inside

thus $$\nabla\cdot\frac{\hat r}{r^2} = 4\pi\delta^3(\vec r)$$

c) $$\nabla\frac{1}{r} = \hat r\frac{\partial}{\partial r}\frac{1}{r} = -\frac{\hat r}{r^2}$$

$$\begin{tikzcd} V \arrow[r, "-\nabla"] & E \arrow[r, "\nabla\cdot"] & \rho/\varepsilon_0 \\ \frac{1}{r} \arrow[r] \arrow[rr, bend right] & \frac{\hat r}{r^2} \arrow[r] & 4\pi\delta^3(\vec r) \end{tikzcd}$$

$$-\nabla^2 V = \rho/\varepsilon_0$$

(Poisson equation)

- Green's functions are the simplest solutions of the Poisson equation

$$G(\vec r,\vec r') \equiv G(\mathscr{r}) = \frac{-1}{4\pi\mathscr{r}} = \nabla^{-2}\delta^3(\vec{\mathscr{r}})$$

  - ~ is a special function which can be used to solve Poisson equation symbolically using the "identity" nature of $\delta^3(\vec r-\vec r') = \delta^3(\vec{\mathscr{r}})$
  - ~ intuitively, it is just the "potential of a point source"

$$\nabla^2 G(\mathscr{r}) = \nabla\cdot\nabla\frac{-1}{4\pi\mathscr{r}} = \nabla\cdot\frac{\hat{\mathscr{r}}}{4\pi\mathscr{r}^2} = \delta^3(\vec{\mathscr{r}}) \qquad \vec{\mathscr{r}} \equiv \vec r - \vec r'$$

let $V = \int_V -G(\mathscr{r})\,\frac{\rho(\vec r')}{\varepsilon_0}\,d\tau'$ (Solution to Poisson's eq.)

$$\nabla^2 V = \int_{V'} -\frac{\rho(\vec r')}{\varepsilon_0}\nabla^2 G(\vec r-\vec r')\,d\tau' = \int_{V'} -\frac{\rho(\vec r')}{\varepsilon_0}\delta^3(\vec r-\vec r')\,d\tau' = -\frac{\rho(\vec r)}{\varepsilon_0}$$

- this generalizes to one of the most powerful methods of solving problems in E&M
  - ~ in QED, Green's functions represent a photon 'propagator'
  - ~ the photon mediates the force between two charges
  - ~ it 'carries' the potential from charge to the other

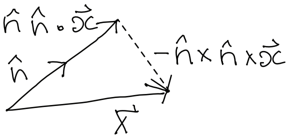
$$U = \int \rho V\,d\tau = \iint \rho G \rho'\,d\tau\,d\tau'$$

Section 1.6 - Helmholtz Theorem

* orthogonal projections $P_{\parallel}$ and $P_{\perp}$: a vector $\vec{n}$ divides the space X into $X_{\parallel n} \oplus X_{\perp n}$

geometric view: dot product $\hat{n} \cdot \vec{x}$ is length of $\vec{x}$ along $\hat{n}$

Projection operator: $P_{\parallel} \equiv \hat{n}\hat{n}\cdot$ acts on x: $P_{\parallel}\vec{x} = \vec{x}_{\parallel} = \hat{n}\hat{n}\cdot\vec{x}$



~ orthogonal projection: $\hat{n}\times$ projects ⊥ to $\hat{n}$ and rotates by 90°

$\hat{X}_{\perp} = -\hat{n}\times(\hat{n}\times\hat{X}) = P_{\perp}\vec{x}$ $\quad P_{\perp} = -\hat{n}\times\hat{n}\times$ $\quad \boxed{P_{\parallel} + P_{\perp} = \hat{n}\hat{n}\cdot - \hat{n}\times\hat{n}\times = I}$

* longtudinal/transverse separation of Laplacian (Hodge decomposition)

$\boxed{\begin{array}{l}\nabla\cdot\vec{F} = \rho \\ \nabla\times\vec{F} = \vec{J}\end{array}}$ ~ is there a solution to these equations for $\vec{F}(r)$ given fixed source fields $\rho(\vec{r})$ and $\vec{J}(\vec{r})$? YES! (compare HW1 #1)

~ proof: $\nabla^2\vec{F} = \nabla\nabla\cdot\vec{F} - \nabla\times\nabla\times\vec{F}$ (longtudinal/transverse components of ∇)

~ formally, $$\vec{F} = -\nabla\underbrace{\left(-\nabla^{-2}\overbrace{\nabla\cdot F}^{\rho}\right)}_{V} + \nabla\times\underbrace{\left(-\nabla^{-2}\overbrace{\nabla\times\vec{F}}^{\vec{J}}\right)}_{\vec{A}}$$

$\rho, \vec{J}$ are SOURCES

$V, \vec{A}$ are POTENTIAL

~ what does ∇^{-2} mean? Note that $-\nabla^2\dfrac{1}{4\pi\mathscr{r}} = \delta^3(\vec{\mathscr{r}})$

~ thus $\nabla^{-2}\delta^3(\vec{\mathscr{r}}) = \dfrac{-1}{4\pi\mathscr{r}} \equiv G(\mathscr{r})$ (see next page) $\quad G = \dfrac{-1}{4\pi\mathscr{r}}$ is Green fn

~ use the δ-identity $\quad \rho(\vec{r}) = \int d\tau'\, \delta^3(\vec{\mathscr{r}})\,\rho(\vec{r}')$

$$V(\vec{r}) \equiv -\nabla^{-2}\rho(\vec{r}) = \int d\tau'\left(-\nabla^{-2}\delta^3(\vec{\mathscr{r}})\right)\rho(\vec{r}') = \int d\tau'\,\frac{\rho(\vec{r}')}{4\pi\mathscr{r}} = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{\mathscr{r}}$$

$$\vec{A}(\vec{r}) \equiv -\nabla^{-2}\vec{J}(\vec{r}) = \int d\tau'\left(-\nabla^{-2}\delta^3(\vec{\mathscr{r}})\right)\vec{J}(\vec{r}') = \int d\tau'\,\frac{\vec{J}(\vec{r}')}{4\pi\mathscr{r}} = \frac{\mu_0}{4\pi}\oint\frac{I\,dl}{\mathscr{r}}$$

~ thus any field can be decomposed into L/T parts $\boxed{\vec{F} = -\nabla V + \nabla\times\vec{A}}$ with $V, \vec{A}$ defined above

SCALAR POTENTIAL V

* Theorem: the following are equivalent definitions of an "irrotational" field:

a) $\nabla\times\vec{F} = \vec{0}$ curl-less

b) $\vec{F} = -\nabla V$ where $V = \displaystyle\int\frac{d\tau'\,\vec{\nabla}\cdot\vec{F}}{4\pi\mathscr{r}}$

c) $V(\vec{r}) = \displaystyle\int_{r_0}^{\vec{r}} -\vec{F}\cdot d\vec{l}$ is independent of path

d) $\oint\vec{F}\cdot d\vec{l} = 0$ for any closed path

* Gauge invariance:

if $\vec{F} = -\nabla V_1$ and also $\vec{F} = -\nabla V_2$

then $\nabla(V_2 - V_1) = 0$ and $V_2 - V_1 = V_0$ is constant

("ground potential")

VECTOR POTENTIAL $\vec{A}$

* Theorem: the following are equivalent definitions of a "solenoidal" field:

a) $\nabla\cdot\vec{F} = 0$ divergence-less

b) $\vec{F} = \nabla\times\vec{A}$ where $\vec{A} = \displaystyle\int\frac{d\tau\,\nabla\times\vec{F}}{4\pi\mathscr{r}}$

c) $? = \displaystyle\int_S \vec{F}\cdot d\vec{a}$ with ∂S fixed is independent of surface

d) $\oint\vec{F}\cdot d\vec{a} = 0$ for any closed surface

* Gauge invariance:

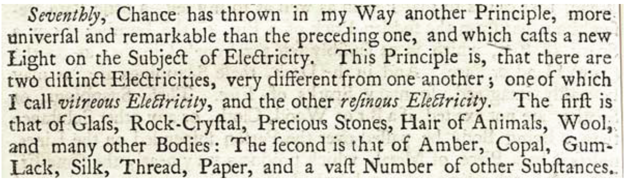
if $\vec{F} = \nabla\wedge\vec{A}_1$ and also $\vec{F} = \nabla\times\vec{A}_2$

then $\nabla\times(A_2 - A_1) = 0$ and $A_2 - A_1 = \nabla\lambda(\vec{r})$

("gauge transformation")

Section 2.1 - Coulomb's Law

* Electric charge (duFay, Franklin)
- ~ +,- equal & opposite (QCD: $r+g+b=0$)
- ~ $e=1.6\times10^{-19}$ C, quantized ($q_n < 2\times10^{-21}\,e$)
- ~ locally conserved (continuity)



Seventhly, Chance has thrown in my Way another Principle, more univerſal and remarkable than the preceding one, and which caſts a new Light on the Subject of Electricity. This Principle is, that there are two diſtinct Electricities, very different from one another; one of which I call *vitreous Electricity*, and the other *reſinous Electricity*. The firſt is that of Glaſs, Rock-Cryſtal, Precious Stones, Hair of Animals, Wool, and many other Bodies: The ſecond is that of Amber, Copal, Gum-Lack, Silk, Thread, Paper, and a vaſt Number of other Subſtances.

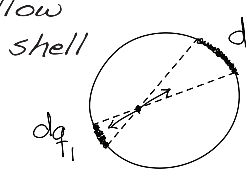
Charles François de Cisternay DuFay, 1734
http://www.sparkmuseum.com/BOOK_DUFAY.HTM

* only for static charge distributions (test charge may move but not sources)

a) Coulomb's law $\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}}$

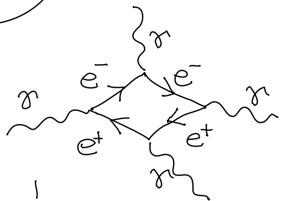
b) Superposition $\vec{F} = \vec{F}_1 + \vec{F}_2 + \ldots$

- ~ Coulomb: torsion balance
- ~ Cavendish: no electric force inside a hollow conducting shell



$dq_2 = \sigma da$
$= \sigma\, d\Omega \cdot r^2$

- ~ Born-Infeld: vacuum polarization violates superposition at the level of $\alpha^2 = \frac{1}{137^2}$



- ~ linear in both q & Q (superposition)
- ~ central force $\vec{\mathscr{r}} \equiv \vec{r} - \vec{r}\,'$
- ~ inverse square (Gauss') law $\frac{1}{\mathscr{r}^2}$
- ~ units: defined in terms of magnetostatics

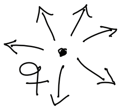
$$\varepsilon_0 = 8.85\times10^{-12}\frac{C^2}{Nm^2} = \frac{1}{\mu_0 c^2}$$

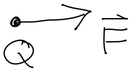
$$1\,C \equiv 1\,A\cdot S \qquad F/_{\ell} = 2\times10^{-7}\,N/_{m}$$

(for parallel wires 1 m apart carrying 1 A each)

- ~ rationalized units to cancel 4π in

$$\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 4\pi\,\delta^3(\vec{\mathscr{r}})$$





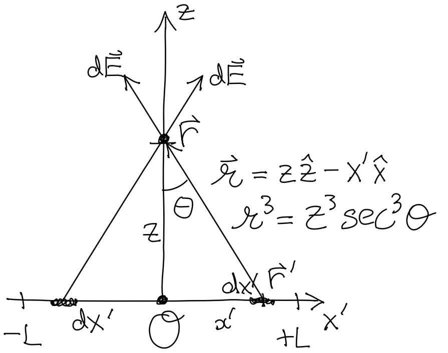
* Electric field
- ~ we want a vector field, but F only at test charge
- ~ action at a distance: the field 'caries' the force from source pt. to field pt.

$$\vec{F} = \underbrace{\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1\hat{\mathscr{r}}_1}{\mathscr{r}_1^2} + \frac{q_2\hat{\mathscr{r}}_2}{\mathscr{r}_2^2} + \ldots\right)}\, Q = Q\vec{E}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i\hat{\mathscr{r}}_i}{\mathscr{r}_i^2} = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{r}\,')d\tau'\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{1}{4\pi\varepsilon_0}\int\frac{dq'\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$dq' \rightarrow q_i = q(\vec{r}_i\,') \text{ or } \lambda(\vec{r}\,')dl' \text{ or } \sigma(\vec{r}\,')da' \text{ or } \rho(\vec{r}\,')d\tau'$$

* Example (Griffiths Ex. 2.1)



$\vec{\mathscr{r}} = z\hat{z} - x'\hat{x}$
$\mathscr{r}^3 = z^3\sec^3\theta$

$$dq' = \lambda dx' = \lambda z\sec^2\theta d\theta$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}2\cdot\int_{x'=0}^{L}\frac{dq'\vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{1}{4\pi\varepsilon_0}\int_0^L\frac{2\lambda dx'\cdot z\hat{z}}{(z^2+x'^2)^{3/2}} + \overset{(sym)}{0\hat{x}}$$

$$= \hat{z}\frac{2\lambda}{4\pi\varepsilon_0 z}\int\frac{\sec^2\theta\, d\theta}{\sec^3\theta}$$

$$= \hat{z}\frac{2\lambda}{4\pi\varepsilon_0 z}\underbrace{\sin\theta}_{x'/\mathscr{r}}\Big|_{x'=0}^{L}$$

$$= \hat{z}\frac{2\lambda}{4\pi\varepsilon_0 z}\frac{L}{\sqrt{z^2+L^2}}$$

$1+\tan^2\theta = \sec^2\theta$
$x' = z\tan\theta$
$dx' = z\sec^2\theta d\theta$
$\mathscr{r}^3 = (z^2+x'^2)^{3/2}$
$= z^3\sec^3\theta$

as $z\to\infty$ $\vec{E} \approx \frac{1}{4\pi\varepsilon_0}\frac{2\lambda L}{z^2}$ as $L\to\infty$ $\vec{E} \approx \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{z}$

Section 2.2 - Divergence and Curl of E

* 5 formulations of electrostatics

$$\begin{tikzcd}[column sep=huge]
\text{Coulomb eq. \& Superposition} & \\
\boxed{\vec{E}=\int\frac{\text{đ}q'\,\hat{\mathscr{r}}}{4\pi\epsilon_0\mathscr{r}^2}} \arrow[dd] \arrow[ddr, leftrightarrow, "\text{Helmholtz}"] & \begin{array}{c}\vec{F}=q\vec{E}\\ W=qV\end{array} \\
\text{Integral field eq's} & \text{Differential field eq's} \\
\boxed{\begin{array}{l}\Phi_E=Q/\epsilon_0\\ \mathcal{E}_E=0 \quad \text{(closed regions)}\end{array}} \arrow[r, shift left=3, rightleftharpoons, "\text{Gauss}"] \arrow[r, shift right=3, rightleftharpoons, "\text{Stokes}"'] \arrow[d, leftrightarrows] & \boxed{\begin{array}{l}\nabla\cdot\vec{E}=\rho/\epsilon_0\\ \nabla\times\vec{E}=\mathbf{0}\end{array}} \arrow[d, leftrightarrows, "\text{Poincaré}"] \\
\mathcal{E}_E=-\Delta V \arrow[r, rightleftharpoons, "\text{FTVC}"] & \vec{E}=-\nabla V \\
\text{Potential} & \text{Poisson eq.} \\
\boxed{V=\int\frac{\text{đ}q'}{4\pi\epsilon_0\mathscr{r}}} \arrow[r, rightleftharpoons, "\text{Laplace}", "\text{Green}"'] & \boxed{\nabla^2V=-\rho/\epsilon_0}
\end{tikzcd}$$

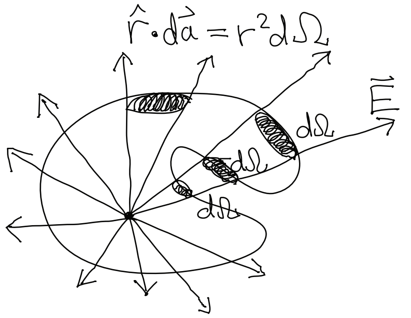
* Gauss' law

~ solid angle

$$d\Omega \equiv \frac{\hat{r}\cdot d\vec{a}}{r^2}$$

~ angle (rad.)

$$d\vec{\theta}=\frac{\hat{r}\times d\vec{l}}{r}$$



~ solid angle of a sphere

$$d\Omega=\sin\theta\, d\theta\, d\phi = -d\cos\theta\, d\phi$$

$$\int\Omega=\int_{\theta=0}^{\pi}-d\cos\theta\cdot\int_{\phi=0}^{2\pi}d\phi = 2\cdot 2\pi = 4\pi$$

~ $\frac{1}{r^2}$ force laws mean there is a const. flux "carrier" field

* Divergence theorem: relationship between differential and integral forms of Gauss' law

$$\Phi_E=\oint_{\partial V}\vec{E}\cdot da = \oint\frac{q\hat{\mathscr{r}}}{4\pi\epsilon_0\mathscr{r}^2}\cdot\hat{\mathscr{r}}\mathscr{r}^2 d\Omega=\frac{q}{\epsilon_0}\rightarrow\int_V\frac{dq}{\epsilon_0}$$

$$\int_V\nabla\cdot\vec{E}\,d\tau=\int_V\rho/\epsilon_0\,d\tau$$

~ since this is true for any volume, we can remove the integral from each side

$$\nabla\cdot\vec{E} \quad = \quad \rho/\epsilon_0$$

~ all of electrostatics comes out of Coulomb's law & superposition principle

~ we use each of the major theorems of vector calculus to rewrite these into five different formulations

- each formulation useful for solving a different kind of problem

~ geometric pictures comes out of schizophrenetic personalities of fields:

* FLOW (Equipotential surfaces)

$\mathcal{E}_E\equiv\int\vec{E}\cdot d\vec{l}$ ~ integral ALONG the field
~ potential = work / charge

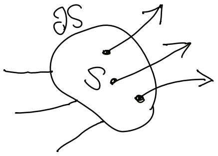
~ $\mathcal{E}_E$ equals # of equipotentials crossed
~ $\Delta\mathcal{E}_E=0$ along an equipotential surface
~ density of surfaces = field strength

* FLUX (Field lines)

$\Phi_E\equiv\int\vec{E}\cdot d\vec{a}$ ~ integral ACROSS the field
~ potential = work / charge

$d\Phi=\vec{E}\cdot d\vec{a}$ = # of lines through area

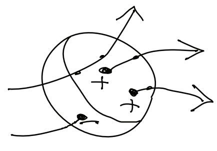
$\vec{E}=\frac{d\Phi}{d\vec{a}}$



~ closed loop

$\int_S d\Phi_E$ = # of lines through loop

~ closed surface



$\int_S d\Phi_E$ = net # of lines out out of surface
= # of charges inside volume

ϵ_0 is unit of proportionality of flux to charge

Section 2.3 - Electric Potential

* two personalities of a vector field: Dr. Jekyl and Mr. Hyde

Flux $= \Phi_E = \int_S \vec{E}\cdot d\vec{a}$ (streamlines) through an area

Flow $= \mathcal{E}_E = \int_P \vec{E}\cdot d\vec{l}$ (equipotentials) downstream

* direct calculation of flow for a point charge

$$\mathcal{E}_E = \int_{\vec{r}=a}^{b} \vec{E}\cdot d\vec{l} = \int_{v'} \frac{dq'}{4\pi\epsilon_0} \int_{\vec{r}=a}^{b} \frac{\hat{\mathscr{r}}\cdot d\vec{l}}{\mathscr{r}^2}$$

note: this is a perfect differential (gradient)

$$\frac{\hat{\mathscr{r}}\cdot d\vec{l}}{\mathscr{r}^2} = \frac{d\mathscr{r}}{\mathscr{r}^2} = d\frac{-1}{\mathscr{r}}$$

$$df = \nabla f \cdot d\vec{l}$$

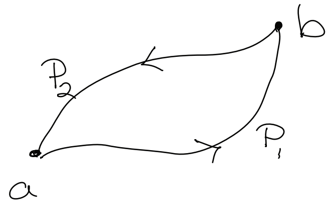
$$\nabla \mathscr{r} = \hat{\mathscr{r}}$$

$$= \int_{v'} \frac{dq'}{\epsilon_0} \frac{1}{4\pi\mathscr{r}}\Big|_{\vec{r}=\vec{r}_a}^{\vec{r}_b} \equiv V(\vec{r})\Big|_a^b$$

~ open path: note that this integral is independent of path

thus $V(\vec{r}) \equiv -\mathcal{E}_E = \int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{l}$ is well-defined

by FTVC: $\Delta V = \int_{\vec{r}_0}^{\vec{r}} \nabla V \cdot d\vec{l}$ so $\boxed{\vec{E} = -\nabla V}$



~ ground potential $V(\vec{r}_0) = 0$ (constant of integration)

~ closed loop (Stokes theorem) $\mathcal{E}_E = \oint_{\partial S} \vec{E}\cdot d\vec{l} = \int_S \nabla\times\vec{E}\cdot d\vec{a} = 0 \iff \boxed{\nabla\times E = 0}$

for any surface S

* Poincaré lemma: if $\vec{E} = -\nabla V$ then $\nabla\times\vec{E} = -\nabla\times\nabla V = 0$

~ converse: if $\nabla\times\vec{E} = 0$ then $\vec{E} = -\nabla V$ so $\boxed{\vec{E} = -\nabla V \iff \nabla\times\vec{E} = 0}$

* Poisson equation $\nabla\cdot\epsilon_0 E = \boxed{-\nabla\cdot\epsilon_0\nabla V = \rho}$ or $\nabla^2 V = \rho/\epsilon_0$

~ next chapter devoted to solving this equation - often easiest for real-life problems

~ a scalar differential equation with boundary conditions on E_n or V

~ inverse (solution) involves: a) the solution for a point charge (Green's function)

$$V(\vec{r}) = \int_{v'} \frac{dq'}{4\pi\epsilon_0} \frac{1}{\mathscr{r}} = \int \frac{dq'}{\epsilon_0} G(\vec{\mathscr{r}}) \quad \text{where} \quad G(\vec{\mathscr{r}}) = \frac{1}{4\pi\mathscr{r}}$$

$$\boxed{\begin{aligned} \nabla^2 G(\vec{\mathscr{r}}) &= \delta^3(\vec{\mathscr{r}}) \\ G(\vec{\mathscr{r}}) &= \nabla^{-2}\delta^3(\vec{\mathscr{r}}) \end{aligned}}$$

$$\nabla^2 G = \nabla\cdot\nabla\frac{1}{4\pi\mathscr{r}} = \nabla\cdot\frac{-\hat{\mathscr{r}}}{4\pi\mathscr{r}^2} = -\delta^3(\vec{\mathscr{r}})$$

b) an arbitrary charge distribution is a sum of point charges (delta functions)

$$\nabla^2 V = \int \frac{dq'}{\epsilon_0}\nabla^2 G = \int_{v'} \frac{\rho(\vec{r}')d\tau'}{\epsilon_0}\delta^3(\vec{\mathscr{r}}) = \frac{\rho(\vec{r})}{\epsilon_0}$$

$$\boxed{\rho(\vec{r}) = \int_{v'} \rho(\vec{r}')d\tau'\,\delta^3(\vec{r}-\vec{r}') = \int_{v'} dq'\,\delta^3(\vec{\mathscr{r}})}$$

going backwards:

$$V = \nabla^{-2}\frac{\rho(\vec{r})}{\epsilon_0} = \int_{v'} \frac{\rho(\vec{r}')d\tau'}{\epsilon_0}\nabla^{-2}\delta^3(\vec{\mathscr{r}}) = \int_{v'} \frac{dq'}{\epsilon_0} G(\vec{\mathscr{r}})$$

~ this is an essential component of the Helmholtz theorem $\boxed{\nabla^2 = \nabla\nabla\cdot - \nabla\times\nabla\times}$

$$\vec{E} = -\nabla\underbrace{\Big(-\nabla^{-2}\overbrace{\nabla\cdot E}^{\rho/\epsilon_0}\Big)}_{V = -\nabla^{-2}\rho/\epsilon_0 = \int_v \frac{dq'}{4\pi\epsilon_0\mathscr{r}}} + \nabla\times\Big(-\nabla^{-2}\overbrace{\nabla\times\vec{E}}^{\vec{0}}\Big) = -\nabla\left(-\nabla^{-2}\rho/\epsilon_0\right) \quad \text{thus} \quad \vec{E} = -\nabla V \iff \nabla\times\vec{E} = 0$$

* derivative chain $\boxed{V \xrightarrow{d} \vec{E} \xrightarrow{d} \rho}$

~ inverting Gauss' law is more tortuous path!

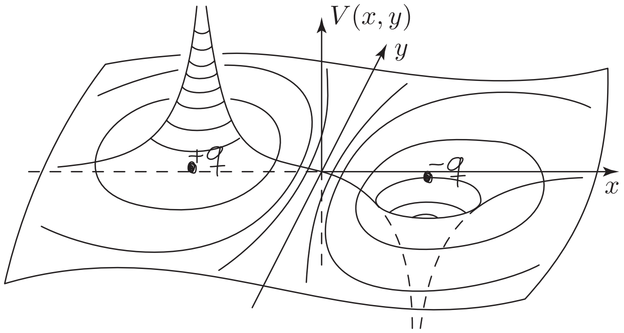
$$\rho \rightarrow V \rightarrow \vec{E} \qquad \vec{E} = -\nabla V = \int \frac{dq'}{4\pi\epsilon_0}\nabla\frac{-1}{\mathscr{r}}$$

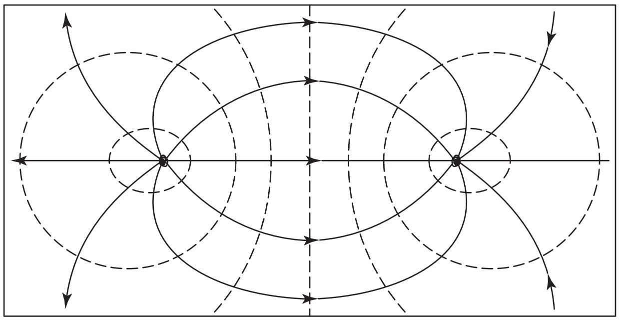
$$\begin{tikzcd}[column sep=huge]
V \arrow[r, shift left, "-\nabla V"] \arrow[rr, bend right=50, leftrightarrow, "\int \vec{E}\cdot d\vec{l}"' near start, "\underset{\int}{\overset{d}{\rightleftharpoons}}" description, "\int\frac{dq'\hat{\mathscr{r}}}{4\pi\epsilon_0\mathscr{r}^2}"' near end] & \vec{E} \arrow[l, shift left, "-\int \vec{E}\cdot d\vec{l}"] \arrow[r, shift left, "\nabla\cdot\epsilon_0\vec{E}"] & \underset{dq'=\rho(\vec{r}')d\tau'}{\rho} \arrow[l, shift left, "\int\frac{dq'\hat{\mathscr{r}}}{4\pi\epsilon_0\mathscr{r}^2}"]
\end{tikzcd}$$

Field Lines and Equipotentials

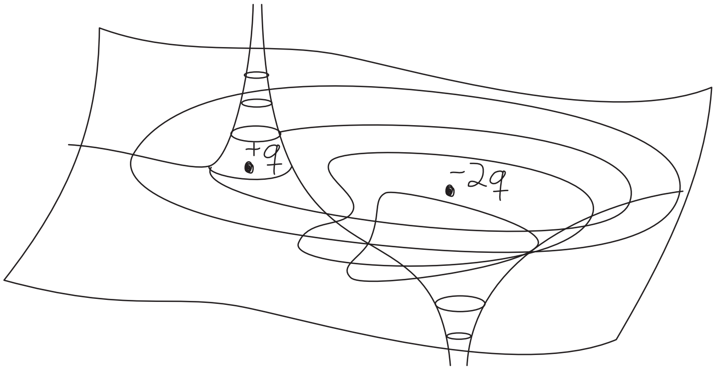
- for   along an equipotential surface:
  fo field lines are normal to equipotential surfaces

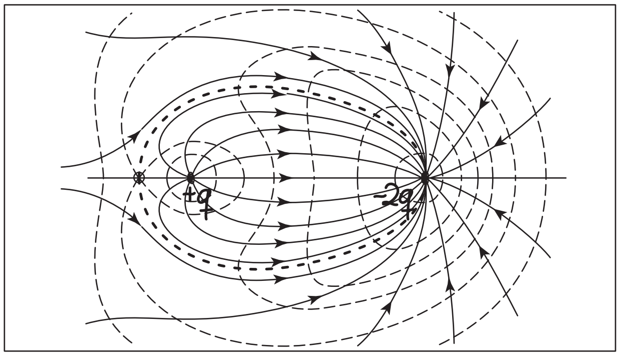
- dipole "two poles" – the word "pole" has two different meanings: (but both are relevant)
  a) opposite (+ vs – , N vs S, bi-polar)
  b) singularity (1/r has a pole at r=0)



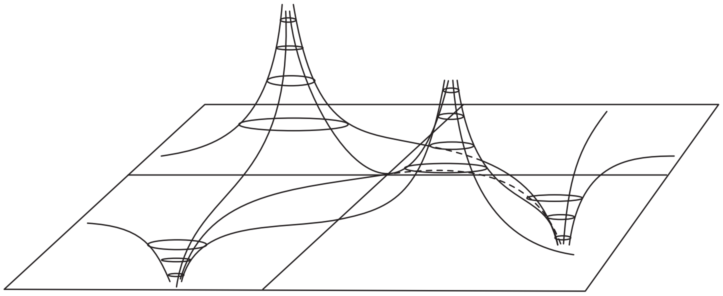


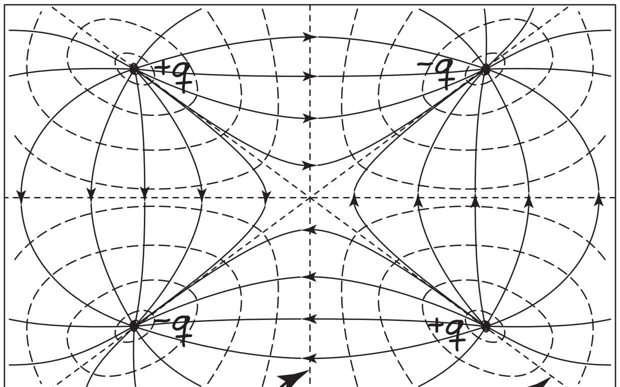
- effective monopole (dominated by –2q far away)





- quadrupole (compare HW3 #2)





sepratrix (potentials)

sepratrix (field lines)

Section 2.4 - Electrostatic Energy

* analogy with gravity

$\vec{F}=q\vec{E}$	$\vec{F}=m\vec{g}$
$W=q\underbrace{Ed}_{\text{potential}=V}$	$W=m\underbrace{gh}_{\text{potential danger}}$

* energy of a point charge in a potential

$$W=\int_a^b \vec{F}\cdot d\vec{l} = -Q\int_a^b \vec{E}\cdot d\vec{l} = Q\Delta V$$

$$W(\vec{r}) = Q\,V(\vec{r}) \qquad V(\infty)\equiv 0$$

* energy of a distribution of charge $q_1, q_2, \ldots$

$$W=\frac{1}{4\pi\varepsilon_0}\left\{q_2\frac{q_1}{r_{12}} + q_3\left(\frac{q_1}{r_{13}}+\frac{q_2}{r_{12}}\right) + q_4\left(\frac{q_1}{r_{14}}+\frac{q_2}{r_{24}}+\frac{q_3}{r_{34}}\right)+\ldots\right\}$$

$$=\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{q_iq_j}{r_{ij}} = \frac{1}{4\pi\varepsilon_0}\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{q_iq_j}{r_{ij}}$$

$$=\frac{1}{2}\sum_{i=1}^{n} q_i \sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_j}{r_{ij}} = \frac{1}{2}\sum_{i=1}^{n} q_i V_i(\vec{r}_i) \qquad W=\frac{1}{2}\sum q_iV_i$$

* continuous version

$$\sum_{i=1}^{n} q_i \to \int dq$$

$$W=\frac{1}{2\varepsilon_0}\int \rho\,\nabla^{-2}\rho\,d\tau$$

$$W=\frac{1}{2}\int \rho V\,d\tau$$

$$W=\frac{\varepsilon_0}{2}\int E^2\,d\tau$$

* energy density stored in the electric field - integration by parts

$$\nabla\cdot V\vec{E} = \nabla V\cdot\vec{E} + V\nabla\cdot E = -\vec{E}\cdot\vec{E} + V\rho/\varepsilon_0$$

$$0=\int_{\partial\infty} d\vec{a}\cdot(V\vec{E}) = \int_{\infty}\nabla\cdot V\vec{E} = \int -E^2 + V\rho/_{\varepsilon_0}\,d\tau$$

$$\boxed{\frac{dW}{d\tau}=\frac{\varepsilon_0E^2}{2}}$$

~ is the energy stored in the field, or in the force between the charges?
~ is the field real, or just a calculational device?
~ if a tree falls in the forest ...

* work does work follow the principle of superposition

~ we know that electric force, electric field, and electric potential do

$$\vec{F}=\vec{F}_1+\vec{F}_2 = q(\vec{E}_1+\vec{E}_2+\ ) = -q\nabla(V_1+V_2+\ldots)$$

~ energy is quadratic in the fields, not linear

$$W_{tot}=\frac{\varepsilon_0}{2}\int E^2 d\tau = \frac{\varepsilon_0}{2}\int E_1^2+E_2^2+2\vec{E}_1\cdot\vec{E}_2\,d\tau$$

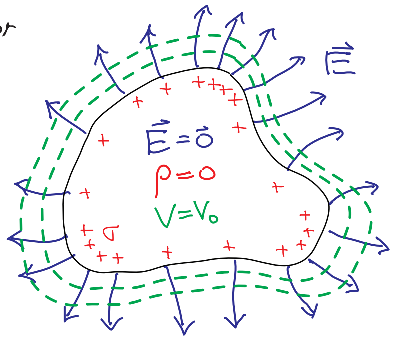
$$= W_1+W_2+\varepsilon_0\int\vec{E}_1\cdot\vec{E}_2\,d\tau$$

~ the cross term is the 'interaction energy' between two charge distributions
(the work required to bring two systems of charge together)

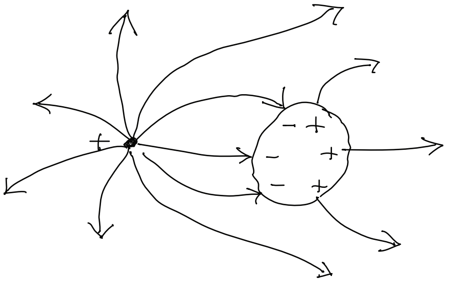
Section 2.5 - Conductors

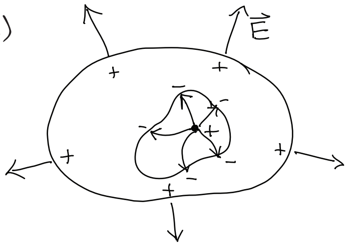
* conductor
  ~ has abundant "free charge", which can move anywhere in the conductor
* types of conductors
  i) metal: conduction band electrons, ~ 1 / atom
  ii) electrolyte: positive & negative ions
* electrical properties of conductors
  i) electric field = 0 inside conductor
  therefore V = constant inside conductor
  ii) electric charge distributes itself
  all on the boundary of the conductor
  iii) electric field is perpendicular to the
  surface just outside the conductor

	inside	outside
ρ	0	σ
$\vec{E}$	$\vec{0}$	$\frac{\sigma\hat{n}}{\varepsilon_0}$
V	V_0	$V_0+\delta$



* induced charges
  ~ free charge will shift around charge on a conductor
  ~ induces opposite charge on near side of conductor
  to cancel out field lines inside the conductor
  ~ Faraday cage: external field lines are shielded
  inside a hollow conductor
  ~ field lines from charge inside a hollow conductor are
  "communicated" outside the conductor by induction
  (as if the charge were distributed on a solid conductor)
  compare: displacement currents, sec. 7.3





* electrostatic pressure
  ~ on the surface: $\vec{F}/A \equiv \vec{f} = \sigma(\cancel{\vec{E}_{patch}} + \vec{E}_{other}) = \frac{1}{2}\sigma(\vec{E}_{inside} + \vec{E}_{outside})$
  ~ for a conductor: $\vec{E}_{inside}=0 \quad \vec{E}_{out} = \sigma/\varepsilon_0 \qquad P = f = \frac{\sigma^2}{2\varepsilon_0} = \frac{\varepsilon}{2}E^2$
  ~ note: electrostatic pressure corresponds to energy density $\quad P \approx w$
  both are part of the stress-energy tensor

Capacitance

* capacitance
- ~ a capacitor is a pair of conductors held at different potentials, stores charge
- ~ electric FLOW from one conductor to the other equals the POTENTIAL difference
- ~ electric FLUX from one conductor to the other is proportional to the CHARGE

$$C = Q/\Delta V = \frac{\varepsilon_0 \Phi_E}{\mathcal{E}_E}$$

$$Q = \int da\,\sigma = \int d\vec{a}\cdot\varepsilon_0\vec{E} = \varepsilon_0 \Phi_E \quad \text{(closed surface)}$$

$$\Delta V = \int d\vec{l}\cdot\vec{E} = \mathcal{E}_E \quad \text{(open path)}$$

- ~ this pattern repeats itself for many other components: resistors, inductors, reluctance (next sememster)

* work formulation

$$W = \frac{1}{2}QV = \frac{1}{2}CV^2 = \int \frac{\varepsilon_0}{2}E^2\,d\tau$$

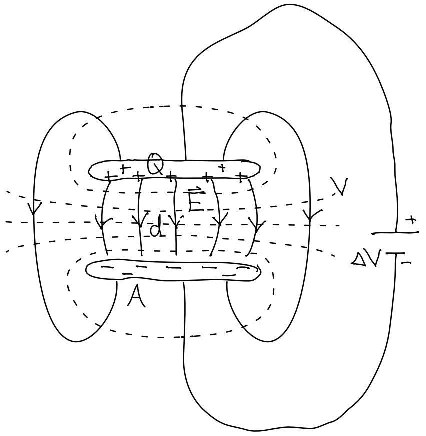
$$= \frac{\varepsilon_0}{2}\,\text{flux}\cdot\text{flow}$$

$$C = \frac{2W}{V^2} = \frac{\varepsilon_0}{V^2}\int E^2\,d\tau = \frac{\varepsilon_0}{2}\,\frac{\text{flux}\cdot\text{flow}}{\text{flow}\cdot\text{flow}}$$

* ex: parallel plates

$$C = \frac{\varepsilon_0 \Phi_E}{\mathcal{E}_E}$$

$$= \frac{\varepsilon_0 E A}{E d} = \frac{\varepsilon_0 A}{d}$$



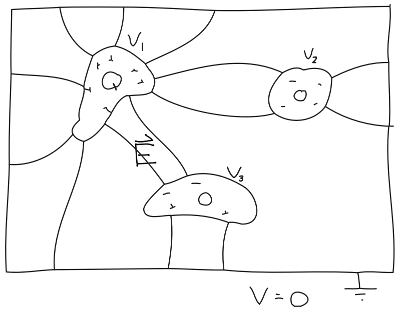
* capacitance matrix
- ~ in a system of conductors, each is at a constant potential
- ~ the potential of each conductor is proportional to the individual charge on each of the conductors
- ~ proportionality expressed as a matrix coefficients of potetial P_{ij} or capacitance matrix C_{ij}

$$V_i = P_{ij}\,Q_j$$

$$Q_i = C_{ij}\,V_j$$

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}$$

$$-\nabla^2 V = \rho/\varepsilon_0 \qquad V(\vec{r}) \propto Q$$



Section 3.1 - Laplace's Equation

* overview: we leared the math (Ch 1) and the physics (Ch2) of electrostatics basically concepts of Phy 232 described in a new sophisticated language
- ~ Ch 3: Boundary Value Problems (BVP) with LaPlace's equation (NEW!)
  - a) method of images b) separation of variables c) multipole expansion
- ~ Ch 4: Dielectric Materials: free and bound charge (more in-depth than Phy 232)

$$\begin{tikzcd}
\chi \arrow[r, "d"] & (V,\vec{A}) \arrow[r, "d"] & (\vec{E},\vec{B}) \arrow[r, "d"] \arrow[d, shift right, "\varepsilon"'] \arrow[dr, leftrightarrow, "\sigma"] & 0 & \\
 & & (\vec{D},\vec{H}) \arrow[u, shift right, "\mu"'] \arrow[r, "d"'] & (\rho,J) \arrow[r, "d"'] & 0
\end{tikzcd}$$

(I) Brute force!

$$\vec{E}=\int\frac{dq'\hat{\imath}}{4\pi\varepsilon_0 \imath^2}$$

(II) Symmetry

$$\Phi_D = Q$$
$$\mathcal{E}_E = 0$$

(III) Elegant but cumbersome

$$\nabla\cdot\vec{D}=\rho \qquad \text{Ch.4}$$
$$\nabla\times\vec{E}=\vec{0}$$

(IV) Refined brute

$$V=\int\frac{dq'}{4\pi\varepsilon_0 \imath}$$

(V) the WORKHORSE!!

$$-\nabla^2 V=\rho/_{\varepsilon} \qquad \text{Ch.3}$$

Equations of electrodyamics:

$\vec{F}=q(\vec{E}+\vec{v}\times\vec{B})$	Lorentz force
$\nabla\cdot\vec{J}+\partial_t\rho=0$	Continuity
$\nabla\cdot\vec{D}=\rho \quad \nabla\times\vec{E}+\partial_t\vec{B}=\vec{0}$	Maxwell electric,
$\nabla\cdot\vec{B}=0 \quad \nabla\times\vec{H}-\partial_t\vec{D}=\vec{J}$	magnetic fields
$\vec{D}=\varepsilon\vec{E} \quad \vec{B}=\mu\vec{H} \quad \vec{J}=\sigma\vec{E}$	Constitution
$\vec{E}=-\nabla V-\partial_t\vec{A} \quad \vec{B}=\nabla\times\vec{A}$	Potentials
$V\to V-\partial_t\lambda \quad \vec{A}\to\vec{A}+\nabla\lambda$	Gauge transform

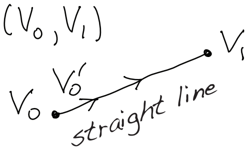
* Classical field equations - many equations, same solution:

Laplace/Poisson:	$\nabla^2V=0 \quad -\nabla\cdot\varepsilon\nabla V=\rho$	~ potentials $(V,\vec{A})$, dielectric ε, permeability μ
Maxwell wave:	$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(V,\vec{A})-\nabla^2(V,\vec{A})=\mu(\rho,\vec{J})$	~ speed of light $c=\frac{1}{\sqrt{\varepsilon\mu}}$, charge/current density $(\rho,\vec{J})$
Heat equation:	$C\frac{\partial T}{\partial t}=k\nabla^2 T$	~ temp T, cond. k, heat $\vec{q}=-k\nabla u$, heat cap. C
Diffusion eq:	$\frac{\partial u}{\partial t}=D\nabla^2 u$	~ concentration u, diffusion D, flow $D\nabla u$
Drumhead wave:	$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}-\nabla^2 u=f$	~ displacement u, speed of sound c, force f
Schrödinger:	$\frac{-\hbar^2}{2m}\nabla^2\psi+V\psi=i\hbar\frac{\partial}{\partial t}\psi$	~ prob amp ψ, mass m, potential V, Planck $\hbar$

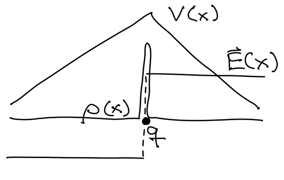
* 1-dimensional Laplace equation $\nabla^2 V=\frac{\partial^2 V}{\partial x^2}=0$

$$\frac{dV}{dx}=\int 0\,dx=a \qquad V=\int a\,dx=ax+b$$

- ~ a,b satisfy boundary conditions (V_0,V_0') or (V_0,V_1)
- ~ mean field: $V(x)=\frac{1}{2}(V(x-a)+V(x+a))$
- ~ no local maxima or minima (stretches tight)



~ charge singularity between two regions:



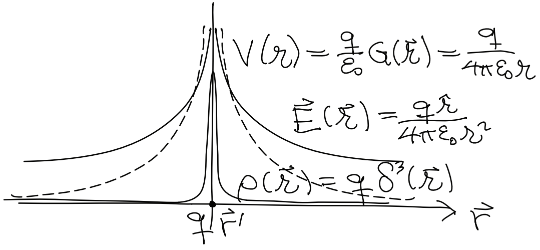
* 2-dimensional Laplace equation $\nabla^2 V=\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0$

- ~ no straighforward solution (method of solution depends on the boundary conditions)
- ~ Partial Differential Equation (elliptic 2nd order)
- ~ chicken & egg: can't solve $\frac{\partial^2 V}{\partial x^2}$ until you know $\frac{\partial^2 V}{\partial y^2}$
- ~ solution of a rubber sheet
- ~ no local extrema -- mean field: $V(\vec{r})=\frac{1}{2\pi R}\oint_{\text{circle}} V\,dl$

* 3-dimensional Laplace equation $\nabla^2 V=\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$

- ~ generalization of 2-d case
- ~ same mean field theorem: $V(\vec{r})=\frac{1}{4\pi R^2}\oint_{\text{sphere}} V\,da$

~ charge singularity between two regions:



Boundary Conditions

* 2nd order PDE's classified in analogy with conic sections: replacing $\frac{\partial}{\partial x}$ with x, etc

a) Elliptic – "spacelike" boundary everywhere (one condition on each boundary point)

eg. Laplace's eq, Poisson's eq. $\nabla^2 V = 0$ $\quad -\nabla\cdot\varepsilon\nabla V = \rho$

b) Hyperbolic – "timelike" (2 initial conditions) and "spacelike" parts of the boundary

eg. Wave equation $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(V,\vec{A}) - \nabla^2(V,\vec{A}) = \mu(\rho,\vec{J})$

c) Parabolic – 1st order in time (1 initial condition)

eg. Heat equation, Diffusion equation $C\frac{\partial T}{\partial t} = k\nabla^2 T$ $\quad \frac{\partial u}{\partial t} = D\nabla^2 u$

* Uniqueness of a BVP (boundary value problem) with Poisson's equation:

if V_1 and V_2 are both solutions of $\nabla^2 V = -\rho/\varepsilon_0$ then let $U = V_1 - V_2$ $\quad \nabla^2 U = 0$

integration by parts: $\nabla\cdot(U\nabla U) = U\,\nabla\cdot\nabla U + \nabla U\cdot\nabla U = U\nabla^2 U + (\nabla U)^2$

in region of interest: $\oint_{\partial V} d\vec{a}\cdot(U\nabla U) = \int_V \nabla\cdot(U\nabla U)\,d\tau = \int_V U\nabla^2 U + (\nabla U)^2\,d\tau$

note that: $\nabla^2 U = 0$ and $(\nabla U)^2 > 0$ always

thus if $\int_{\partial V} d\vec{a}\cdot U\nabla U = \int_{\partial V} da\,\underbrace{U}_{(a)}\,\underbrace{\frac{\partial U}{\partial n}}_{(b)} = 0$ then $\int(\nabla U)^2 d\tau = 0 \Rightarrow U = 0$ everywhere

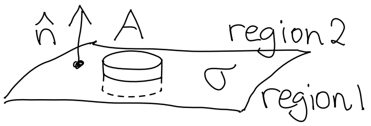
a) Dirichlet boundary condition: $U = 0$ – specify potential $V_1 = V_2$ on boundary

b) Neuman boundary condition: $\frac{\partial U}{\partial n} = 0$ – specify flux $\frac{\partial V_1}{\partial n} = \frac{\partial V_2}{\partial n}$ on boundary

* Continuity boundary conditions – on the interface between two materials

Flux:

$\vec{D} \equiv \varepsilon\vec{E}$
(shorthand for now)



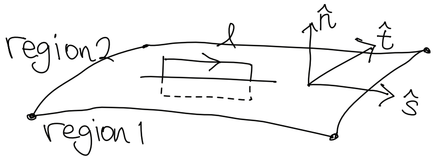
$$\Phi = \oint_{\partial V}\vec{D}\cdot d\vec{a} = \int_V \sigma\,da = Q$$

$$\hat{n}\cdot(\vec{D}_2 - \vec{D}_1)\,A = \sigma\cdot A$$

$$\hat{n}\cdot(\vec{D}_2 - \vec{D}_1) = \sigma$$

$$-\frac{\partial V_2}{\partial n} + \frac{\partial V_1}{\partial n} = \sigma/\varepsilon_0$$

Flow:



$\hat{s}\times\hat{t} = \hat{n}$

$$\oint_{\partial S}\vec{E}\cdot d\vec{l} = \int_S \nabla\times\vec{E}\cdot d\vec{a}$$

$$\hat{s}\cdot(\vec{E}_2 - \vec{E}_1)\,l = \hat{t}\cdot\nabla\times\vec{E}\;l\,w = 0$$

$$\hat{n}\times(\vec{E}_2 - \vec{E}_1) = 0$$

$$V_2 = V_1$$

* the same results obtained by integrating field equations across the normal

$$\nabla\cdot\vec{D} = \rho/\varepsilon_0$$

$$\int_-^+ dn\left(\frac{\partial D_n}{\partial n} + \frac{\partial D_s}{\partial s} + \frac{\partial D_t}{\partial t}\right) = \int_-^+ dn\,\sigma\,\delta(n)$$

$$\int dD_n = \hat{n}\cdot\Delta\vec{D} = \sigma$$

$$\nabla\times\vec{E} = \vec{K}_e\,\delta(n) \qquad \nabla\times\vec{E} = \begin{vmatrix}\hat{s} & \hat{t} & \hat{n}\\ \partial_s & \partial_t & \partial_n\\ E_s & E_t & E_n\end{vmatrix}$$

$$\int_-^+ dn\left(\hat{t}\,\frac{\partial E_s}{\partial n} - \hat{s}\,\frac{\partial E_t}{\partial n}\right) = \int_-^+ dn\,\vec{K}_e\,\delta(n)$$

$$\hat{n}\times\Delta\vec{E} = \vec{K}_e = 0$$

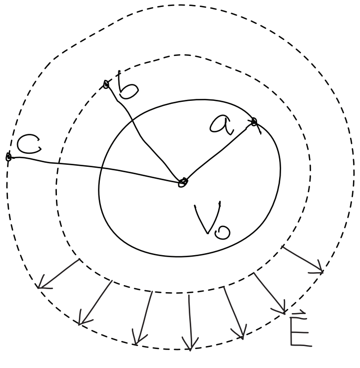
~ opposite boundary conditions for magnetic fields: $\hat{n}\cdot\Delta\vec{B} = 0$ $\quad \hat{n}\times\Delta\vec{H} = \vec{K}$

Section 3.2 - Method of Images

* concept: in a region $\mathcal{R}$, $V(\vec{r})$ depends ONLY on the boundary of V at $\partial\mathcal{R}$
~ it doesn't matter how it was created, or where charge is outside $\mathcal{R}$
~ more than one charge distribution can generate the same $V(\vec{r})$ inside $\mathcal{R}$

* Example 1: $V=V_0$ inside a constant sphere of radius a

$V=\frac{q}{4\pi\varepsilon_0 r}$ for a point charge at the origin, OR on the outside of a uniform spherical shell of total charge q



i) $q_0 = 4\pi\varepsilon_0 a \cdot V_0$ on a sphere of radius 'a' is one solution

ii) $q_1 = 4\pi\varepsilon_0 b\ V_0$ on a sphere of radius 'b' another solution

iii) how about $+q_2$ at radius 'b' and $-q_2$ at radius 'c'?

for $r>c$, $V = \frac{+q_2}{4\pi\varepsilon_0 r} + \frac{-q_2}{4\pi\varepsilon_0 r} = 0$ $\quad E=0$ for $r>c$

and for $r<b$ $V=\frac{q_2}{4\pi\varepsilon_0 b} - \frac{q_2}{4\pi\varepsilon_0 c} = V_0 = \frac{q_0}{4\pi\varepsilon_0 a}$, the equivalent charge of (i)

so $q_2 = \frac{V_0}{4\pi\varepsilon_0}\left(\frac{1}{b}-\frac{1}{c}\right)^{-1}$ for example, if $\begin{matrix} b=a \\ c=2a \end{matrix}$ then $q_2 = 2q_0$

in the case, the nonzero E-field between b and c, and builds up the potential at a

* Example 2: dipole: point charge $+q$ at $z=d$ and $-q$ at $z=-d$

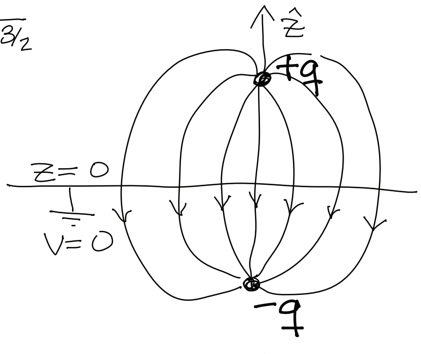
$$V(z) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{-q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

~ note that $V(z=0)=0$ so we can form a boundary value problem for $z>0$, $V(z=0)=0$ with the same solution!

~ induced surface charge: $\sigma = \varepsilon_0 E_n = -\varepsilon_0 \frac{\partial V}{\partial n} = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}$
total induced charge:

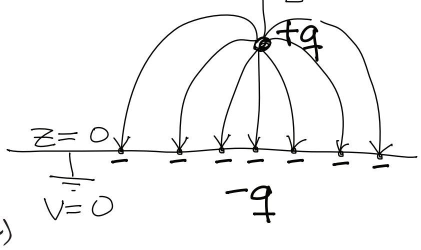
$$Q = \int \sigma\, da = \int_{\phi=0}^{2\pi}\int_{s=0}^{\infty} \sigma\, s\, ds\, d\phi = \frac{-2\pi q d}{2\pi}\int_{s=0}^{\infty}(s^2+d^2)^{-3/2} s\, ds$$

$$= -qd\int_{s=0}^{\infty} u^{-3/2}\,\tfrac{1}{2}du = qd\cdot -u^{-1/2}\Big|_{s=0}^{\infty} = -q \qquad \text{let } \begin{matrix} u=s^2+d^2 \\ du=2s\,ds \end{matrix}$$



~ force on the charge:

$$\vec{F} = q\vec{E} = -q\nabla V = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)^2}\hat{z}$$



~ energy in the system: $W = \frac{1}{2}(W_0) = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{q^2}{2d}$

this is only half the value of dipole problem, because the induced charge is brought into zero potential (no work)

Section 3.3.1 - Separation of Variables (Cartesian)

* goal: solve Lapalce's equation (a single PDE) by converting it into one ODE per variable
method: separate the equation into separate terms in x,y,z
start by factoring the solution $V(x,y,z) = X(x)\,Y(y)\,Z(z)$
trick: if $f(x) = g(y)$ where $f(x)$ is independent of y and $g(y)$ is independent of x
then they must both constant
endgame: form the most possible general solution as a linear combination of
all possible products of solutions in each variable.
Solve for unique values of the coefficients using the boundary conditions
analogy: the set of all solutions forms a vector space
the basis vectors are independent individual solutions

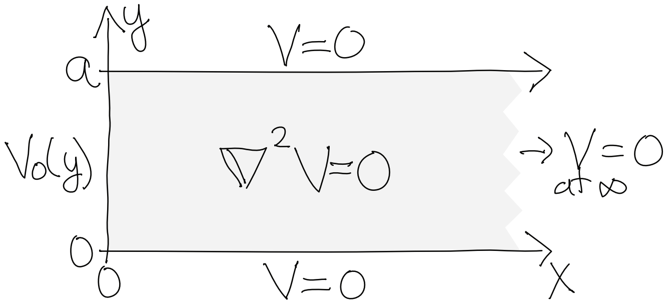
* Example: semi-infinite strip with non-zero voltage at one end

$$V(x,y) = X(x)\cdot Y(y)$$

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{f(x)=k^2} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{g(y)=-k^2} = 0$$

$$X'' - k^2X = 0 \qquad Y'' + k^2 Y = 0$$

$$X = Ae^{kx} + Be^{-kx} \qquad Y = C\sin(ky) + D\cos(ky)$$



~ boundary conditions (BC):

1) $V(\infty,y)=0 \Rightarrow A=0$

2) $V(x,0)=0 \Rightarrow D=0$

3) $V(x,a)=0 \Rightarrow \sin(ka)=0 \quad k_n=\frac{n\pi}{a}$

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-k_nx}\sin(k_ny)$$

4) $V_0(y) = \sum_{n=1}^{\infty} c_n \sin(k_n y)$

$$\int_0^a \sin(K_my)\,V_0(y)\,dy = \sum_{n=1}^{\infty} c_n \int_0^a \sin(k_ny)\cdot\sin(k_my)\,dy$$

$$= \sum_{n=1}^{\infty} c_n \int_0^a \cos\left(\frac{(n-m)\pi}{a}y\right) - \cos\left(\frac{(n+m)\pi}{a}y\right) dy.$$

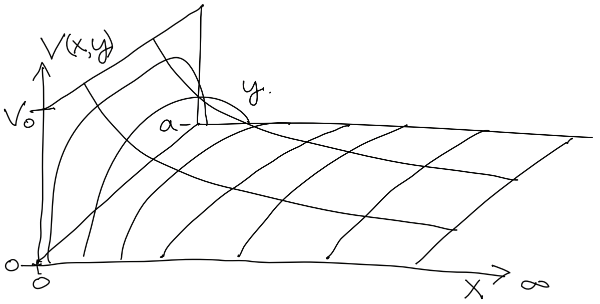
$$= \sum_{n=1}^{\infty} c_n \frac{a}{2}\delta_{nm} \qquad = \frac{a}{2}c_m$$

~ if $V_0(y) = \text{const} = V_0$ then

$$C_n = \frac{2}{a}\int_0^a \sin\left(\frac{n\pi y}{a}\right)V_0\,dy = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{4V_0}{n\pi} & \text{if } n \text{ odd}\end{cases}$$

$$V(x,y) = \sum_{n=1,3,5\ldots}^{\infty} \frac{4V_0}{n\pi}\, e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

(Fourier decomposition)



* Vector Analogy:

$$\hat{e}_i\cdot\vec{V} = \hat{e}_i\cdot(V_j\,\hat{e}_j)$$
$$= V_j\,\delta_{ij} = V_i$$

$$\hat{x}\cdot(a\hat{x}+b\hat{y}+c\hat{z}) = a$$
$$\hat{y}\cdot(a\hat{x}+b\hat{y}+c\hat{z}) = b$$
$$\hat{z}\cdot(a\hat{x}+b\hat{y}+c\hat{z}) = c$$

$$\phi_n(x) = \sin(k_nx) \qquad V(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)$$
$$\langle\phi_n|\phi_m\rangle = \int_0^a \sin(k_nx)\cdot\sin(k_mx)\,dx = \frac{a}{2}\delta_{nm}$$
$$C_m = \langle\phi_m(x)|V(x)\rangle \Big/ \frac{a}{2}.$$

Section 3.3.2 - Separation of Variables (Spherical)

* same technique as in rectangular coordinates
- the differential equations are more complex, but we only solve them once
- boudnary conditions are of two types
  - a) radial - external boundary condition - treated in the same way as cartesian
  - b) angular - internal to the problem - almost always have the same solution

* key principles:
- separation of variables $V(r, \theta, \phi) = R(r)\,\Theta(\theta)\,\Phi(\phi)$
- orthogonality of $\Theta(\theta) = P_l(\cos\theta)$
- boundary conditions $r \to 0,\ r = a,\ r \to \infty$

* separation of variables - slight twist: solve one eigenvalue at a time

$$\nabla^2 V(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}V + \underbrace{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}V + \frac{1}{r^2\sin^2\theta}\overbrace{\frac{\partial^2}{\partial\phi^2}V}^{-m^2V}}_{-l(l+1)V} = -\rho/_{\varepsilon_0} = 0$$

RADIAL EQUATION

$$\frac{d}{dr}r^2\frac{d}{dr}R(r) = l(l+1)R(r)$$

let $R(r) = r^\alpha$ $\quad \alpha(\alpha+1) = l(l+1)$

$\alpha = l\ ,\ -(l+1)$

$$R(r) = Ar^l + Br^{-l-1}$$

POLAR EQUATION (m=0)

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}\Theta(\theta) = -l(l+1)\,\Theta(\theta)$$

let $x = \cos(\theta)$ $\quad dx = -\sin\theta\, d\theta$

$\Theta(\theta) = P_l(x)$ $\quad \sin\theta\, d\theta\, d\phi \to -dx\, d\phi$

$$\frac{d}{dx}(1-x^2)\frac{d}{dx}P_l(x) + l(l+1)P_l(x) = 0$$

$\Theta(\theta) = P_l(x) = P_l(\cos\theta)$; $Q_l(\cos\theta)$ diverges

AZIMUTHAL EQ.

$$\frac{d^2}{d\phi^2}\Phi = -m^2\Phi$$

$\Phi(\phi) = e^{im\phi}$

if $m=0$ then

$\Phi(\phi) = const$

* general solution

$\nabla^2 V = 0$ $$V(r,\theta) = \sum_{l-0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

* boundary conditions

i) at $r=0$, $\frac{1}{r^{l+1}} \to \infty$ so $B_l = 0$

ii) at $r=\infty$, $r^l \to \infty$ so $A_l = 0$

iii) at $r=a$, (1) $V_0(\theta) = V(a,\theta) = \sum_{l=0}^{\infty}\left(A_l a^l + \frac{B_l}{a^{l+1}}\right)P_l(\cos\theta)$ $\qquad E_{ext} = E_0\hat{x} = -\nabla(-r^1\cos\theta)$

(2) $\frac{\partial V_0}{\partial r}(\theta) = \frac{\partial V}{\partial r}(a,\theta) = \sum_{l=0}^{\infty}\left(lA_l a^{l+1} - (l+1)\frac{B_l}{a^{l+2}}\right)P_l(\cos\theta)$

surface boundary at the interface between two regions with surface charge σ

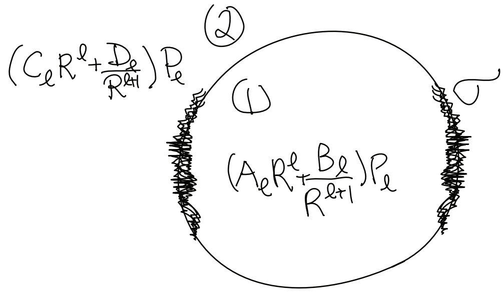
$$\begin{matrix}\nabla\cdot\varepsilon_0\vec{E} = \rho \\ \nabla\times\vec{E} = 0\end{matrix} \Rightarrow \begin{matrix}\hat{n}\cdot(\vec{E}_2-\vec{E}_1) = \sigma/_{\varepsilon_0} \\ \hat{n}\times(\vec{E}_2-\vec{E}_1) = 0\end{matrix} \quad \begin{matrix}E_{2n}-E_{1n} = \sigma/_{\varepsilon_0} \\ E_{2t}-E_{1t} = 0\end{matrix} \Rightarrow \begin{matrix}V_1'(a) - V_2'(a) = \sigma/_{\varepsilon_0} \\ V_1(a) = V_2(a)\end{matrix}$$

* properties of the Legendre polynomials
- Rodrigues formula $P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l(x^2-1)^l \quad l = 0, 1, 2, \ldots$
- orthogonality

$$\langle P_l | P_{l'}\rangle \equiv \int_{-1}^{1}P_l(x)P_{l'}(x)\,dx = \int_0^{\pi}P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta = \begin{cases}0 & \text{if } l' \neq l \\ \frac{2}{2l+1} & \text{if } l' = l\end{cases}$$

- this is only one independent solution
  the other solutions $Q(x)$ blows up at the N&S poles $(\theta = 0, 2\pi)$
  and doesn't satisfy continuity boundary conditions

Problem 3.9



$4 \times \infty$ unknowns
4 B.C.'s.

* spherical shell of charge $\quad \sigma = \sigma_0 \sin^2\theta$

inside region: $\quad V_1(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$

outside region: $\quad V_2(r,\theta) = \sum_{\ell=0}^{\infty} \left(C_\ell r^\ell + \frac{D_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$

boundary conditions:

i) $V_1(0,\theta)$ finite $\quad \Longrightarrow \quad B_\ell = 0$

ii) $V_2(\infty,\theta)$ finite $\quad \Longrightarrow \quad C_\ell = 0$ (let $C_0 = 0$ also)

iii) $V_1(R,\theta) = V_2(R,\theta) \quad \sum_{\ell=0}^{\infty} (A_\ell R^\ell + 0) P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} \left(0 + \frac{D_\ell}{R^{\ell+1}}\right) P_\ell$

$$\sum_{\ell=0}^{\infty} \left(A_\ell R^\ell - \frac{D_\ell}{R^{\ell+1}}\right) P_\ell(\cos\theta) = 0 \quad \Longrightarrow \quad D_\ell = A_\ell R^{2\ell+1}$$

iv) $E_{2n} - E_{1n} = \sigma/\varepsilon_0$

$$-\frac{\partial V_2}{\partial r}\Big|_R + \frac{\partial V_1}{\partial r}\Big|_R = \frac{\sigma}{\varepsilon_0} = \frac{\sigma_0}{\varepsilon_0}\sin^2\theta$$

$$\sum_{\ell=0}^{\infty} \left(D_\ell \frac{(\ell+1)}{R^{\ell+2}} + A_\ell \cdot \ell R^{\ell-1}\right) P_\ell(\cos\theta) = \frac{\sigma_0}{\varepsilon_0}\sin^2\theta$$

$$\sin^2\theta = 1 - \cos^2\theta$$
$$= -\cos^2\theta + \tfrac{1}{3} + \tfrac{2}{3}$$
$$= -\tfrac{2}{3}P_2(\cos\theta) + \tfrac{2}{3}P_0(\cos\theta)$$

* $\displaystyle \sum_{\ell=0}^{\infty} A_\ell (2\ell+1) R^{\ell-1} \cdot P_\ell(\cos\theta) = \frac{\sigma_0}{\varepsilon_0}\sin^2\theta$

$$(A_0 R^{-1})P_0 + (A_1 \cdot 3R^0)P_1 + (A_2 \cdot 5R)P_2 + \ldots = \left(\frac{\sigma_0}{\varepsilon_0}\frac{2}{3}\right)P_0 + 0 + \left(\frac{\sigma_0}{\varepsilon_0}\cdot\frac{-2}{3}\right)P_2 + \ldots$$

$$A_0 = \frac{2\sigma_0}{3\varepsilon_0 R} \qquad A_1 = 0 \qquad A_2 = \frac{-2\sigma_0 R}{15\varepsilon_0}$$

solutions: inside $\quad V_1 = \frac{+2\sigma_0}{3\varepsilon_0}\left(\frac{1}{R} - \frac{r^2}{5R^3}\frac{1}{2}(3\cos^2\theta - 1)\right) \qquad V_1 = V_2 \quad @\ r = R$

outside $\quad V_2 = \frac{+2}{3}\frac{\sigma_0}{\varepsilon_0}\left(\frac{1}{R} - \frac{R^2}{5r^3}\frac{1}{2}(3\cos^2\theta - 1)\right) \qquad -V_2' + V_1' = \sigma/\varepsilon_0 \quad @\ r = R$

alternate solution of B.C. iv (use integrals to extract components like in Section 3.2.1)

* $\displaystyle \int_{\theta=0}^{\pi} P_0(\cos\theta)\cdot\sin^2\theta \ \sin\theta\, d\theta = \int_0^{\pi} \sin^3\theta\, d\theta = \frac{4}{3} \qquad \int_{\theta=0}^{\pi} P_0(\cos\theta)\cdot P_0(\cos\theta)\ \sin\theta\, d\theta = \int_0^{\pi} \sin\theta\, d\theta = \frac{2}{1}$

$$\int_0^{\pi} P_1(\cos\theta)\cdot\sin^2\theta\ \sin\theta\, d\theta = \int_0^{\pi} \cos\theta\cdot\sin^3\theta\, d\theta = 0 \qquad \int_0^{\pi} P_1(\cos\theta)\cdot P_1(\cos\theta)\ \sin\theta\, d\theta = \int_0^{\pi} \cos^2\theta\cdot\sin\theta\, d\theta = \frac{2}{3}$$

$$\int_0^{\pi} P_2(\cos\theta)\cdot\sin^2\theta\ \sin\theta\, d\theta = \int_0^{\pi} \frac{1}{2}(3\cos^2\theta - 1)\cdot\sin^3\theta\, d\theta = \frac{-4}{15} \qquad \int_0^{\pi} P_2(\cos\theta)\cdot P_2(\cos\theta)\ \sin\theta\, d\theta = \int_0^{\pi} \frac{1}{4}(3\cos^2\theta - 1)^2\cdot\sin\theta\, d\theta = \frac{2}{5}$$

Section 3.4 - Multipoles

* binomial expansion

$$(a+b)^0 = 1$$
$$(a+b)^1 = a + b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = \binom{4}{0}a^4b^0 + \binom{4}{1}a^3b^1 + \binom{4}{2}a^2b^2 + \binom{4}{3}a^1b^3 + \binom{4}{4}a^0b^4$$

* Pascal's triangle

$$\begin{array}{l|ccccccccc}
n=0 & & & & & 1 & & & & \\
n=1 & & & & 1 & & 1 & & & \\
n=2 & & & 1 & & 2 & & 1 & & \\
n=3 & & 1 & & 3 & & 3 & & 1 & \\
n=4 & 1 & & 4 & & 6 & & 4 & & 1 \\
\hline
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}$$

~ general form

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdots\ k}$$

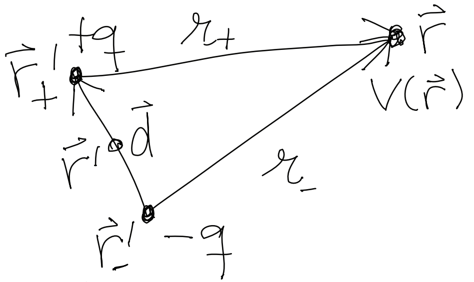
~ if $n \to \alpha$ (any real number), then the series does not terminate unless $\alpha = 0, 1, 2, \ldots$

$$(1+x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{1\cdot 2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{1\cdot 2\cdot 3}x^3 + \cdots$$

~ example: $\displaystyle \frac{1}{1-x} = 1-(-x) + \frac{-1\cdot -2}{1\cdot 2}(-x)^2 + \frac{-1\cdot -2\cdot -3}{1\cdot 2\cdot 3}(-x)^3 + \ldots$

$$= 1 + x + x^2 + x^3 + \ldots \qquad \text{for radius of convergence } |x| < 1$$

* 2-pole expansion



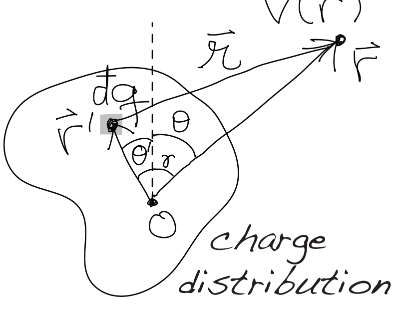
$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right)$$

$$\vec{\mathscr{r}}_\pm = \vec{\mathscr{r}} \mp \tfrac{1}{2}\vec{d} \qquad \mathscr{r}_\pm^2 = \mathscr{r}^2 \mp \mathscr{r} d\cos\theta + \tfrac{1}{4}d^2$$

$$\frac{1}{\mathscr{r}_\pm} \cong \frac{1}{\mathscr{r}}\left(1 \mp \frac{d}{\mathscr{r}}\cos\theta\right)^{-\frac{1}{2}} \qquad \approx \frac{1}{\mathscr{r}}\left(1 \pm \frac{d}{2\mathscr{r}}\cos\theta + \ldots\right)$$

$$V(\vec{r}) = \frac{qd\cos\theta}{4\pi\varepsilon_0 \mathscr{r}^2} = \frac{\vec{p}\cdot\vec{\mathscr{r}}}{4\pi\varepsilon_0\mathscr{r}^3} \qquad \boxed{\vec{p} = q\vec{d}} \ \text{electric dipole moment}$$

* general axial-symmetric multipole expansion



$$\mathscr{r}^2 = (\vec{r}-\vec{r}\,')^2 = r^2\left(1 - 2\frac{r'}{r}\cos\gamma + \left(\frac{r'}{r}\right)^2\right) \equiv r^2(1+\varepsilon)$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\varepsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \ldots\right))$$

$$= \frac{1}{r}\left(1 + \frac{r'}{r}\cos\gamma + \frac{r'^2}{r^2}\frac{1}{2}(3\cos^2\gamma - 1) + \frac{r'^3}{r^3}\frac{1}{2}(5\cos^3\gamma - 3\cos\gamma\right) + \ldots$$

$$= \frac{1}{r}\left(P_0(\cos\gamma) + \frac{r'}{r}P_1(\cos\gamma) + \frac{r'^2}{r^2}P_2(\cos\gamma) + \ldots\right. \Rightarrow \boxed{\sum_{l=0}^{\infty}\frac{r'^l}{r^{l+1}}P_l(\cos\theta')\,P_l(\cos\theta)}$$

(addition formula for azimuthally symmetry)

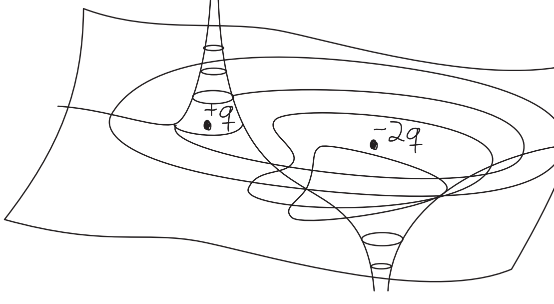
$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}P_l(\cos\theta)\underbrace{\int dq'\, r'^l P_l(\cos\theta')}_{Q^{(l)}_{\text{int}}}$$

multipole potential $\qquad Q^{(l)}_{\text{int}}$ electric multipole (monopole, dipole, quadrupole)

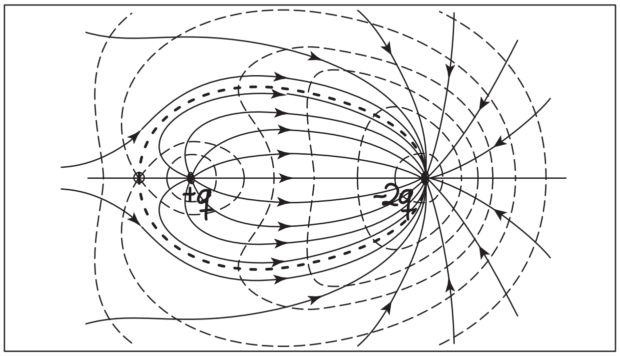
~ $Q^{(l)}_{\text{int}}$ are coefficients of the general solution of Laplace equation in spherical coords

* *monopole* $V(\vec{r}) \approx \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int dq' = \frac{q}{4\pi\varepsilon_0 r}$

$q \equiv \int dq'$

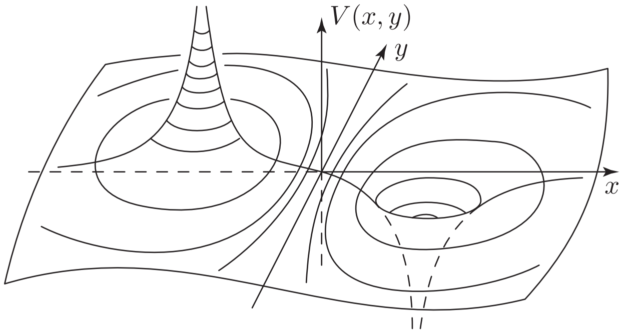


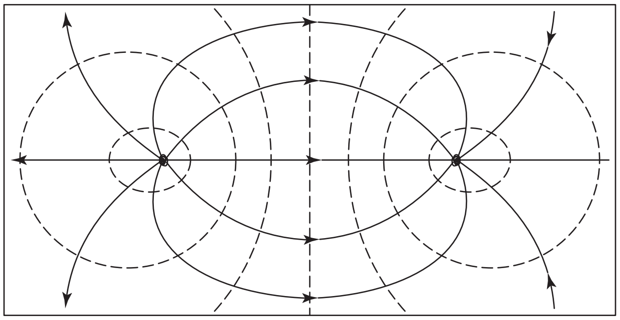
effective monopole
(dominated by -2q
far from the origin)



* *dipole* $V_1(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^2}\int dq' r' \cos\gamma = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3}$ $\vec{p} = \int dq' \vec{r}\,'$ $\vec{r}\cdot\vec{r}\,' = r r' \cos\gamma$

if $q = \int dq' = 0$ then $T_{\vec{a}}[\vec{p}] = \int dq' (\vec{r}\,' - \vec{a}) = \int dq' \vec{r}\,' - \vec{a}\overset{0}{\cancel{\int dq'}} = \vec{p}$





* *quadrupole*

$$V_2(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^3}\int dq' r'^2 \frac{1}{2}(3\cos^2\gamma - 1) = \frac{1}{4\pi\varepsilon_0 r^5}\int dq' \frac{1}{2}(3(\vec{r}\cdot\vec{r}\,')^2 - r^2)$$ $Q_{xx} + Q_{yy} + Q_{zz} = 0$

$$= \frac{\frac{1}{2}\vec{r}\cdot\overleftrightarrow{Q}\cdot\vec{r}}{4\pi\varepsilon_0 r^5}$$

$$\overleftrightarrow{Q} = \int dq' (3\vec{r}\,'\vec{r}\,' - I r'^2) = \int dq' \begin{pmatrix} 3x'^2 - r'^2 & 3x'y' & 3x'z' \\ 3y'x' & 3y'^2 - r'^2 & 3y'z^2 \\ 3z'x' & 3z'y' & 3z'^2 - r'^2 \end{pmatrix}$$

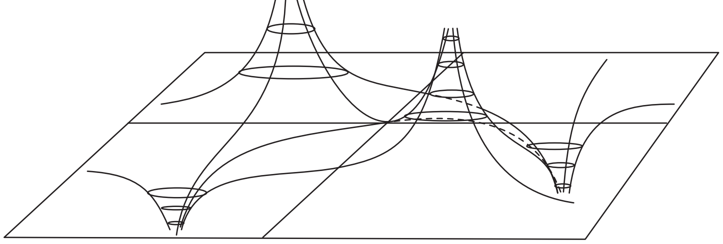
$$Q_{ij} = \int dq' (3 r_i' r_j' - \delta_{ij} r'^2)$$

$$T_{\vec{a}}[\overleftrightarrow{Q}] = \int dq' 3(\vec{r}\,' - \vec{a})(\vec{r} - \vec{a}) - (\vec{r}\,' - \vec{a})^2 I$$

$$= \int dq' (3\vec{r}\,'\vec{r}\,' - \vec{r}\,'^2 I) - 3(\vec{r}\,'\vec{a} + \vec{a}\vec{r}\,' - \vec{a}\vec{a}) + (2\vec{r}\cdot\vec{a} + \vec{a}^2) I$$

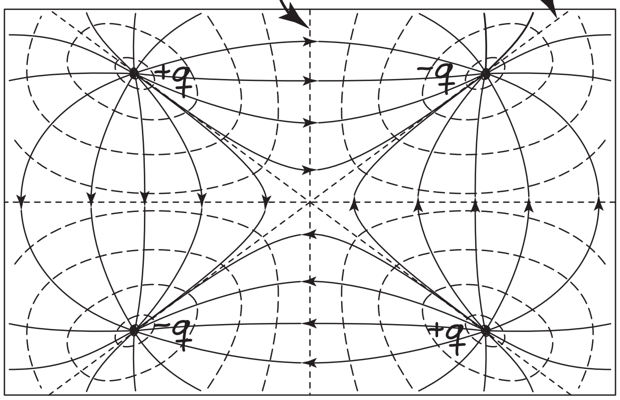
$$= \overleftrightarrow{Q}_0 - [3(\vec{p}\vec{a} + \vec{a}\vec{p}) - 2\vec{p}\cdot\vec{a} I] + [3\vec{a}\vec{a} - \vec{a}^2 I] q$$

$$\overleftrightarrow{Q} = 3\sum_i \vec{p}_i\vec{a}_i + \vec{a}_i\vec{p}_i - 2\vec{p}_i\cdot\vec{a}_i I$$ dipoles $\vec{p}_i$
at positions $\vec{a}_i$



sepratrix
(potentials)

sepratrix
(field lines)



Section 3.4 - Multipoles (continued)

* spherical solutions $V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$ solving Laplace equation $\nabla^2 V = 0$

* internal multipole $Q^{(l)}_{int} = B_l$

$r \to \infty$: $V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty} Q^{(l)}_{int} \frac{1}{r^{l+1}} P_l(\cos\theta)$

$r' \to 0$: $Q^{(l)}_{int} = \int dq'\, r'^l P_l(\cos\theta')$

	$l=0$	1	2	3
V	$1/r$	$1/r^2$	$1/r^3$	$1/r^4$
E	$1/r^2$	$1/r^3$	$1/r^4$	$1/r^5$

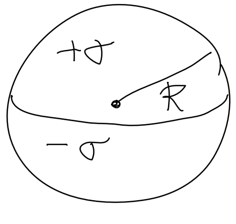
* external multipole $Q^{(l)}_{ext} = A_l$

$r \to 0$: $V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty} Q^{(l)}_{ext}\, r^l P_l(\cos\theta)$

$r' \to \infty$: $Q^{(l)}_{ext} = \int dq' \frac{1}{r'^{l+1}} P_l(\cos\theta')$

	$l=0$	1	2	3
V	const	r	r^2	r^3
E	—	const	r	r^2

* example: calculate the dipole moment of two oppositely charge hemispheres



$\sigma = \frac{q}{2\pi R^2}$

$\vec{p} = \int dq'\, \vec{r}'$ $\quad p_x = p_y = 0$

$$p_z = \int_{\theta=0}^{\pi} \sigma' da' \cdot z' = \int_{x=-1}^{1} \sigma \cdot 2\pi R^2 dx\, Rx$$

$$= \int_{x=-1}^{0} \frac{-q}{2\pi R^2} 2\pi R^2 dx\, Rx + \int_{x=0}^{1} \frac{q}{2\pi R^2} 2\pi R^2 dx\, Rx = qR\left[\int_{-1}^{0} x dx + \int_{0}^{1} x dx\right] = qR$$

* example: internal and external moments of a 4-pole $+q$ @ (a,a) $\quad -q$ @ $(a,-a)$
$-q$ @ $(-a,a)$ $\quad +q$ @ $(-a,-a)$

$q = \sum_i q_i = q + q - q - q = 0$

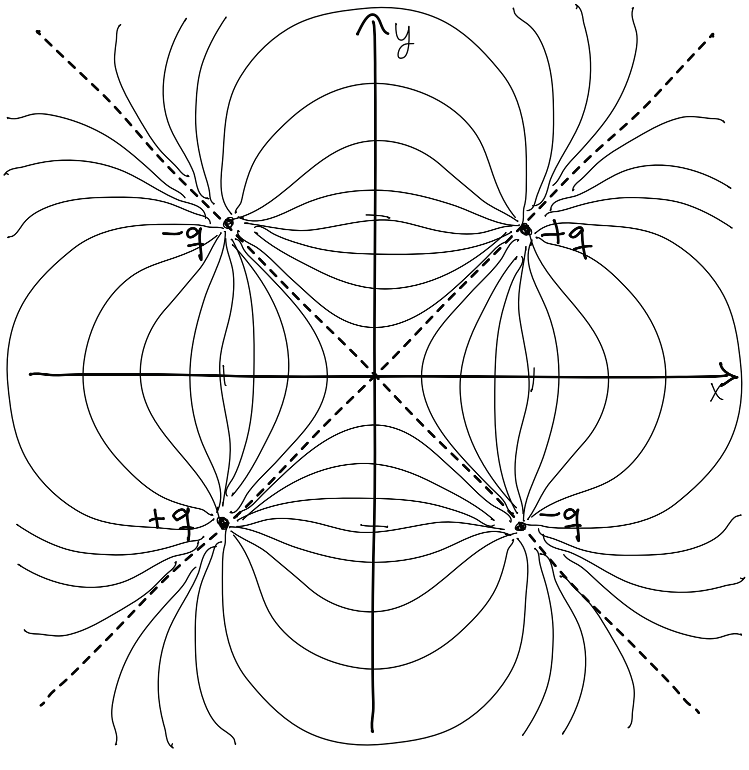
$\vec{p} = \sum_i q_i \vec{r}_i = q(a,a) - q(-a,a)$
$\quad - q(a,-a) + q(-a,-a) = \vec{0}$

$Q_{zx} = \sum_i 3 q_i z_i x_i = 0 = Q_{zy}$

$Q_{xy} = \sum_i 3 q_i x_i y_i$
$\quad = +3q \cdot a \cdot a - 3q(-a)(a)$
$\quad - 3q(a)(-a) + 3q(-a)(-a) = 4qa^2$

$Q_{xx} = \sum_i q_i (3x_i'^2 - r'^2)$
$\quad = q(3a^2 - 2a^2) - q(3a^2 - 2a^2)$
$\quad - q(3a^2 - 2a^2) + q(3a^2 - 2a^2) = 0$

$Q_{zz} = \sum_i q_i (3z_i^2 - r_i^2)$
$\quad = (q - q - q + q)(0 - 2a^2) = 0$



* *electric field of a dipole* $V = \frac{p\cos\theta}{4\pi_0\varepsilon r^2}$

$$\vec{E} = \frac{-\partial V}{\partial r}\hat{r} + \frac{-1}{r}\frac{\partial V}{\partial\theta}\hat{\theta} = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}\hat{r} + \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}\hat{\theta}$$

$$= \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$= \frac{p}{4\pi\varepsilon_0 r^3}\left(3\cos\theta\,\hat{r} - \hat{z}\right) = 3\,\frac{\vec{p}\cdot\hat{r}\,\hat{r} - \vec{p}}{4\pi\varepsilon_0 r^3}$$

Section 4.1 - Polarization

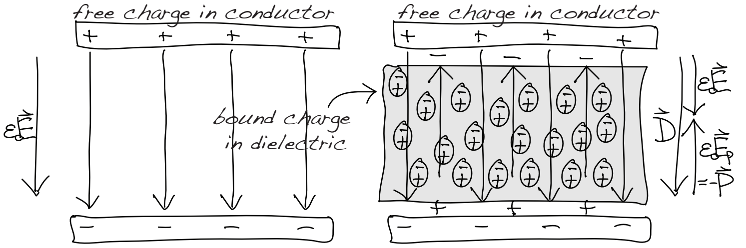
- * Overview
  - ~ Ch3: Poisson/Laplace equation more powerful than integrating the field/potential over charge distributions (for example, don't need to know the charge on a conductor)
  - ~ Ch4: Extend formalism to dielectic media (deal with charges in individual atoms)

$$\nabla\cdot\vec{E} = \rho/\varepsilon_0 \qquad \xrightarrow[\varepsilon_0\vec{E}\to\vec{D}]{\varepsilon_0\to\varepsilon} \qquad \nabla\cdot\vec{D} = \rho_f$$

$$\nabla\times\vec{E} = \vec{0} \qquad\qquad\qquad \nabla\times\vec{E} = 0$$

- * Dielectrics
  - ~ charge is bound to neutral atoms
  - ~ not free, but can still polarize
  - ~ either stretching or rotating

* example: parallel plate capacitor

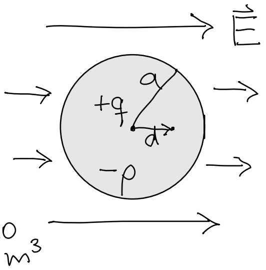


- * Induced dipoles
  - ~ field stretches charge apart in atom
  - ~ atomic polarizability tensor

$$\vec{p} = \tilde{\alpha}\vec{E} \qquad \vec{p} = \alpha_\perp \vec{E}_\perp + \alpha_\parallel \vec{E}_\parallel$$

$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

* example: nucleus in a cloud of charge



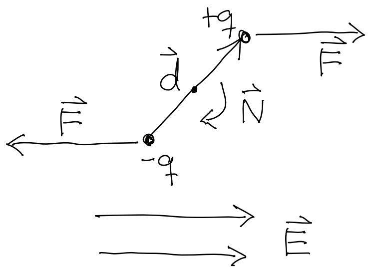
$$E_e = \frac{1}{4\pi\varepsilon_0}\frac{qd}{a^3}$$

$$p = qd = 4\pi\varepsilon_0 a^3 E$$

$$\alpha = 4\pi\varepsilon_0 a^3 = 3\varepsilon_0 v$$

$$\frac{\alpha}{4\pi\varepsilon_0} \approx a^3 \approx 1\text{Å}^3 \approx 10^{-30}\,m^3$$

* Dipole in an electric field



$$\vec{N} = \vec{r}_+\times\vec{F}_+ + \vec{r}_-\times\vec{F}_-$$
$$= \vec{d}/2\times q\vec{E} + \frac{-\vec{d}}{2}\times -q\vec{E}$$
$$= q\vec{d}\times\vec{E} = \vec{p}\times\vec{E}$$

$$U = \int N\,d\theta = \int pE\sin\theta$$
$$= -pE\cos\theta = -\vec{p}\cdot\vec{E}$$

$$\vec{F} = \vec{F}_+ + \vec{F}_- = q(\vec{E}_+ - \vec{E}_-)$$
$$= qd\,\frac{\Delta E}{d} = (\vec{p}\cdot\nabla)\vec{E}$$
$$= \nabla(\vec{p}\cdot\vec{E}) \quad \text{if } \vec{p} \text{ const.}$$

Section 4.2 - Polarization Fields

* review: dipole moment, polarization, forces, dielectrics

$$\vec{p} = \int dq' \vec{r}' = q\vec{d} \qquad \vec{N} = \vec{p}\times\vec{E}$$

$$d\vec{p} = \vec{P}\,d\tau \qquad U = -\vec{p}\cdot\vec{E}$$

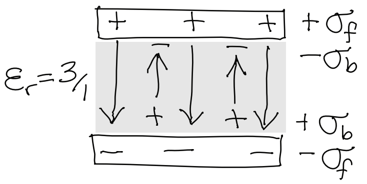
$$\vec{F} = q\vec{E} \qquad \vec{F} = (\vec{p}\cdot\nabla)\vec{E}$$

~ does $(2q)\vec{d} = q(2\vec{d})$?

~ what about $\vec{d}_1 = -\vec{d}_2$?

~ force on a quadrupole?

$\mathcal{E}$, $\mathcal{D}$, $\mathcal{P}$ all point down!



$$\varepsilon_0 E = D - P = 3 - 2 = 1$$

* electric potential from polarization: bound charge

$$V = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}'} \frac{d\vec{p}'\cdot\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}'} \vec{P}'d\tau'\cdot\nabla'\frac{1}{\mathscr{r}}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\oint_{\partial\mathcal{V}} \frac{\vec{P}'\cdot\hat{n}'\,da'}{\mathscr{r}} + \int_{\mathcal{V}} \frac{-\nabla'\cdot P'\,d\tau'}{\mathscr{r}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\oint_{\partial\mathcal{V}} \frac{\sigma_b'\,da'}{\mathscr{r}} + \int_{\mathcal{V}} \frac{\rho_b'\,d\tau'}{\mathscr{r}}\right]$$

$$\nabla\frac{1}{r} = \frac{-1}{r^2}\nabla r = \frac{-\hat{r}}{r^2}$$

$$\nabla\frac{1}{\mathscr{r}} = \frac{-1}{\mathscr{r}}\nabla\mathscr{r} = \frac{-\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\nabla'\frac{1}{\mathscr{r}} = \frac{d}{d\vec{r}'}\frac{1}{\mathscr{r}} = -\nabla\frac{1}{\mathscr{r}}$$

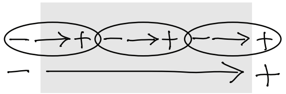
$$\nabla\cdot\frac{\vec{P}}{\mathscr{r}} = \frac{\nabla\cdot\vec{P}}{\mathscr{r}} + \vec{P}\cdot\nabla\frac{1}{\mathscr{r}}$$

bound charge $\boxed{\sigma_b = \vec{P}\cdot\hat{n}}$ $\boxed{\rho_b = -\nabla\cdot\vec{P}}$

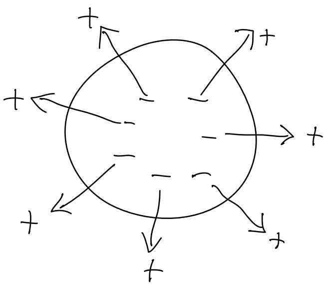
~ uncancelled charge from overlapping dipoles in the polarized (dielectric) medium

* physical interpretation of bound charge

~ polarization forms "dipole chains"



$$\sigma_b = \frac{q_b}{A} = \frac{q\vec{d}\cdot\hat{n}}{Ad} = \vec{P}\cdot\hat{n}$$



~ divergence finds lone charge at end of each chain

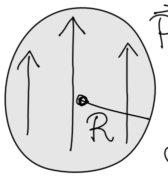
$$\int_{\mathcal{V}}\rho_b\,d\tau = -\oint_{\partial\mathcal{V}}\vec{P}\cdot d\vec{a} = -\int_{\mathcal{V}}\nabla\cdot P\,d\tau \qquad \rho_b = -\nabla\cdot\vec{P}$$

~ what fluxes have we considered so far?

~ what are the similarities and differences between $\vec{E}$ and $\vec{P}$?

~ why are they so similar?

* example 4.2 - bound charge and fields of a sphere with constant polarization $\vec{P}$



$$\vec{P} = P\hat{z}$$

$$\rho_b = -\nabla\cdot\vec{P} = 0$$

$$\sigma_b = \vec{P}\cdot\hat{n} = P\cos\theta$$

$$V_1 = \sum_\ell a_\ell\left(\frac{r}{R}\right)^\ell P_\ell(x)$$

$$V_2 = \sum_\ell a_\ell\left(\frac{R}{r}\right)^{\ell+1} P_\ell(x)$$

$V_1 = V_2$:

$$\sum_\ell a_\ell P_\ell(x) = \sum_\ell a_\ell P_\ell(x)$$

$V_1' - V_2' = \sigma/\varepsilon_0$:

$$\sum_\ell a_\ell\left(\frac{\ell}{R}\left(\frac{r}{R}\right)^{\ell-1} + \frac{\ell+1}{R}\left(\frac{R}{r}\right)^{\ell+2}\right)P_\ell(x) = P/\varepsilon_0\cos\theta$$

$$\sum_\ell a_\ell\frac{2\ell+1}{R}P_\ell(x) = P/\varepsilon_0\,P_1(x) \qquad a_\ell = \frac{PR}{3\varepsilon_0}\delta_{\ell 1}$$

$$V_1 = \frac{P}{3\varepsilon_0}r\cos\theta = \frac{Pz}{3\varepsilon_0} \qquad \vec{E}_1 = -\nabla V_1 = \frac{-\vec{P}}{3\varepsilon_0}$$

$$V_2 = \frac{PR}{3\varepsilon_0}\frac{R^2}{r^2}\cos\theta = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3} \quad \text{where} \quad \vec{p} = \frac{4}{3}\pi R^3\vec{P}$$

$$\vec{E}_2 = -\nabla V_2 = \frac{\vec{p}}{4\pi\varepsilon_0 r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) = \frac{3\vec{p}\cdot\hat{r}\hat{r} - \vec{p}}{4\pi\varepsilon_0 r^3}$$

Section 4.3 - Electric Displacement $\vec{D}$

* reviews: parallels between E and P
~ what are the units of $\varepsilon_0\vec{E}$? $\vec{P}$?
~ both are vector fields (functions of position)
~ the field lines (flux) are associated with charge
(Dr. Jekyll or Mr. Hyde ??)
~ the two fields are related: E induces P in a dielectric

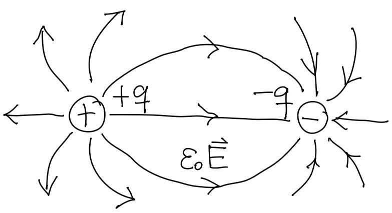
$$\begin{array}{llll} \Phi_{\varepsilon_0 E}=Q & \nabla\cdot\varepsilon_0\vec{E}=\rho & \hat{n}\cdot\Delta\varepsilon_0\vec{E}=\sigma & \text{total charge}\\ (+)\ \Phi_P={}^-Q_b & (+)\ \nabla\cdot\vec{P}=-\rho_b & (+)\ \hat{n}\cdot\Delta\vec{P}=-\sigma_b & -\text{ bound charge}\\ \hline \Phi_D=Q_f & \nabla\cdot\vec{D}=\rho_f & \hat{n}\cdot\Delta\vec{D}=\sigma_f & =\text{ free charge}\\ & & D_2^{\perp}-D_1^{\perp}=\sigma_f & \end{array}$$

* new field: D = "electric displacement"
~ defined by the "constitutive equation": $\boxed{\vec{D}=\varepsilon_0\vec{E}+\vec{P}}$
~ associated with the free charge:
lines of D flux go from (+) to (−) free charge
~ iterative cycle:
a) free charge generates E
b) E causes P, diplaced bound charge ρ_b σ_b
c) the field from bound charge modifies E
~ direct calculation procedure with D
a) calculate D directly from free charge only
b) obtain P from D using consititutive relation
c) the electric field is: $\varepsilon_0\vec{E}=\vec{D}-\vec{P}$

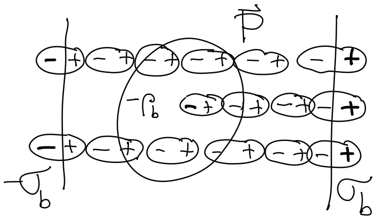
* differences between $\varepsilon_0 E$, P, and D:
~ equipotentials associated with force $\vec{F}=q\vec{E}$
only for the electric field
~ ρ generates E, but P induces ρ_b
~ $\mathcal{E}_{\Gamma}=0$ $\quad\nabla\times\vec{E}=\vec{0}$ $\quad\hat{n}\times\Delta\vec{E}=\vec{0}$
~ $\vec{E}=\int\frac{dq'\,\hat{\mathscr{r}}}{4\pi\varepsilon_0\mathscr{r}^2}$ $\quad\vec{E}_b=\int\frac{dq'_b\,\hat{\mathscr{r}}}{4\pi\varepsilon_0\mathscr{r}^2}$ $\quad\vec{E}_f=\int\frac{dq'_f\,\hat{\mathscr{r}}}{4\pi\varepsilon_0\mathscr{r}^2}$
note: not P or D in these formulas!

* you need both $\nabla\cdot\vec{D}=\rho_f$ and $\nabla\times\vec{E}=0$ to solve!

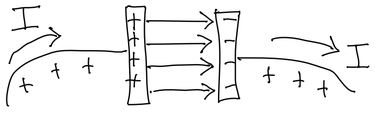
Electric field "E"



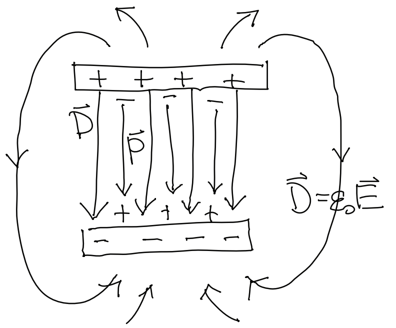
Polarization "P"



"Displacement current"
(Maxwell) $\vec{J}_d=\frac{\partial\vec{D}}{\partial t}$



$$I_d=\int\vec{J}_d\cdot d\vec{a}=\int\frac{\partial\vec{D}}{\partial t}\cdot d\vec{a}=\frac{\partial\Phi_D}{\partial t}$$



$\varepsilon_0 E=D-P$ inside
$\varepsilon_0 E=D$ outside

Section 4.4.1 - Linear Dielectrics

* going from polarizability (α) to susceptibility (χ_e)

~ atoms: $\vec{p} = \alpha \vec{E}$

~ material: $\vec{P} = \varepsilon_0 \chi_e \vec{E} = \frac{\Delta \vec{p}}{\Delta \tau} = \frac{\Delta n}{\Delta \tau} \vec{p} = N \alpha \vec{E}$ $\boxed{\varepsilon_0 \chi_e \cong N\alpha}$ (See HW9 for refinements)

* material properties:

~ linear: χ_e is independent of field magnitude $|\vec{E}|$

~ isotropic: χ_e is a scalar (independent of direction $\hat{E}$)

~ homogeneous: χ_e is independent of position

~ nonisotropic material:

$\vec{P} = \varepsilon_0 \overleftrightarrow{\chi}_e \vec{E}$ (like $\overleftrightarrow{\alpha}$)

$$\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \begin{pmatrix} \chi_{xx} & \chi_{xy} & \chi_{xz} \\ \chi_x & \chi_{yy} & \chi_{yz} \\ \chi_x & \chi_{zy} & \chi_{zz} \end{pmatrix}$$

* permeability: absolute $\varepsilon = \varepsilon_0 \varepsilon_r$, relative $\varepsilon_r = K$ (dielectric const.)

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \underbrace{(1+\chi_e)}_{\varepsilon_r} \vec{E} = \underbrace{\varepsilon_0 \varepsilon_r}_{\varepsilon} \vec{E} = \varepsilon \vec{E}$$

~ constitutive eq: $\boxed{\vec{D} = \varepsilon \vec{E}}$

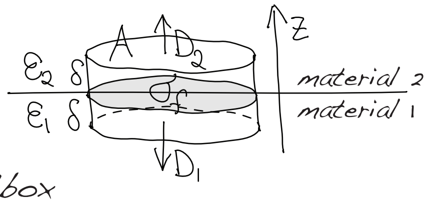
~ property of the material: $\varepsilon_r = 1 + \chi_e = \varepsilon / \varepsilon_0$

others: $\vec{B} = \mu \vec{H}$ $\vec{J} = \sigma \vec{E}$

* continuity boundary conditions

a) FLUX

$\boxed{\nabla \cdot \vec{D} = \rho_f}$

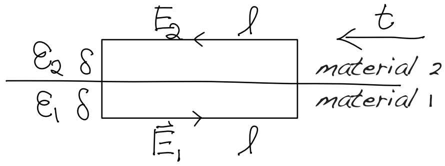


~ Gaussian pillbox

$$\Phi_D = \hat{z} \cdot D_2 A - \hat{z} \cdot D_1 A = \sigma_f A = Q_f$$

b) FLOW

$\boxed{\nabla \times \vec{E} = 0}$



~ Amperian loop

$$\mathcal{E}_E = \hat{t} \cdot \vec{E}_2 l - \hat{t} \cdot \vec{E}_1 l = 0$$

~ Integration of $\nabla \cdot \vec{D} = \rho_f$ across boundary

$$\lim_{\delta \to 0} \int_{-\delta}^{\delta} dz \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) = \int_{-\delta}^{\delta} \sigma_f \delta(z-z')\, dz$$

$$\int_{-\delta}^{\delta} dD_z = \boxed{\hat{n} \cdot \Delta \vec{D} = \sigma_f} \quad \boxed{-\varepsilon_2 \frac{\partial V_2}{\partial n} + \varepsilon_1 \frac{\partial V_2}{\partial n} = \sigma_f}$$

~ Integration of $\nabla \times \vec{E} = 0$ across boundary

$$\lim_{\delta \to 0} \int_{-\delta}^{\delta} dz\, \hat{x} \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) + \hat{y} \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) + \hat{z} \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) = 0$$

$$= \int_{-\delta}^{\delta} \hat{x}\, dE_y - \hat{y}\, dE_x = \boxed{\hat{n} \times \Delta \vec{E} = 0} \quad \boxed{V_2 = V_1}$$

~ the only difference in dielectric boundary value problems is $\varepsilon_1, \varepsilon_2$ in boundary cond.

* example 4.7: dielectric ball in electric field

$$V_1 = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-l-1} \right) P_l(\cos\theta)$$

$$V_2 = \sum_{l=0}^{\infty} \left(C_l r^l + D_l r^{-l-1} \right) P_l(\cos\theta)$$

$\lim r \to 0$ $V_1(r) \neq \infty$ $B_l = 0$

$\lim r \to \infty$ $V_2(r) = -E_0 r \cos\theta$ $C_l = -E_0 \delta_{l1}$

$V_1(R) = V_2(R)$ $A_l R^l = C_l R^l + D_l R^{-l-1}$

$-\varepsilon_2 V_2'(R) + \varepsilon_1 V_1'(R) = \sigma_f = 0$

$-\varepsilon_2 \left(C_l \cdot l R^{l-1} + D_l(-l-1) R^{-l-2} \right) + \varepsilon_1 \left(A_l l R^{l-1} \right) = 0$

if $l \neq 1$ $D_l = A_l R^{2l+1}$ $D_l = A_l = 0$

if $l = 1$ $A_1 = -E_0 + D_1 R^{-3}$

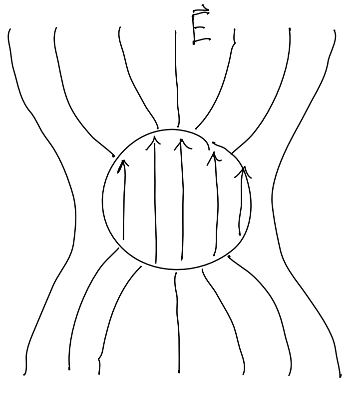
$- \varepsilon_2 (-E_0 - 2 D_1 R^{-3}) + \varepsilon_1 A_1 = 0$

$-\varepsilon_2 (-E_0 - 2(A_1 + E_0)) + \varepsilon_1 A_1 = 0$

$3\varepsilon_2 E_0 + (\varepsilon_1 + 2\varepsilon_2) A_1 = 0$

$$A_1 = \frac{-3\varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} E_0$$

if $\varepsilon_1 = \varepsilon_r \varepsilon_2$ $A_1 = \frac{-3}{\varepsilon + 2} E_0$



Section 4.4.3 - Energy in Dielectric Systems

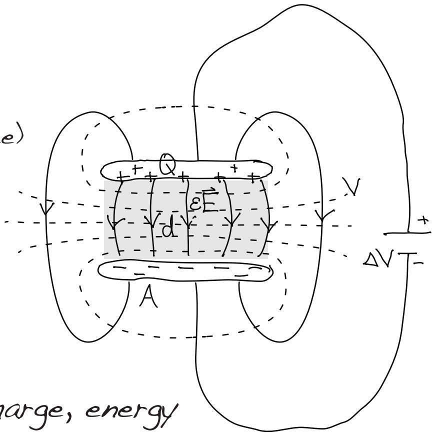
* capacitance = flux/flow

$$C = \frac{Q}{\Delta V} = \frac{\Phi_D}{\mathcal{E}_F} = \frac{\varepsilon\Phi_E}{\mathcal{E}_E} \approx \frac{\varepsilon A}{d}$$

$Q = \int d\vec{a}\cdot\vec{D} = \Phi_D$ (closed surface)

$\Delta V = \int d\vec{l}\cdot\vec{E} = \mathcal{E}_E$ (open path)

$\varepsilon_0\vec{E} \to \vec{D}$ $\qquad \varepsilon_0 \to \varepsilon = \varepsilon_0\varepsilon_r$



* energy in a capacitor $\qquad W = \frac{1}{2}CV^2 = \frac{1}{2}QV$

~ where does the 1/2 come from?

~ ε_r (dielectric const) enhancement factor of capacitance, charge, energy

* energy in the electric field = flux x flow $\qquad W = \frac{\varepsilon_0}{2}\int E^2 d\tau \to \frac{\varepsilon}{2}\int E^2 d\tau = \frac{1}{2}\int \vec{D}\cdot\vec{E}\, d\tau$

~ proof: $V \xrightarrow{d} \vec{E}$ or $\vec{D} \xrightarrow{d} \rho_f$

$$\Delta W = \int \Delta\rho_f V\, d\tau = \int(\nabla\cdot\Delta\vec{D}) \quad d\tau = \oint d\vec{a}\cdot(\Delta\vec{D}\, V) - \int\Delta\vec{D}\cdot\nabla V\, d\tau = \int\Delta\vec{D}\cdot\vec{E}\, d\tau$$

~ for a linear dielectric (linear materials),

$$\Delta W = \int\varepsilon\,\Delta\vec{E}\cdot\vec{E}\, d\tau = \int\frac{\varepsilon}{2}\Delta E^2 d\tau = \Delta\frac{1}{2}\int\vec{D}\cdot\vec{E}\, d\tau$$

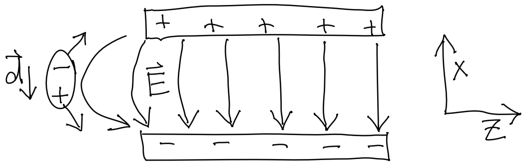
$$\boxed{W = \frac{1}{2}\int\vec{D}\cdot\vec{E}\, d\tau}$$

* forces on dielectrics

~ force of a fringe field on a dipole

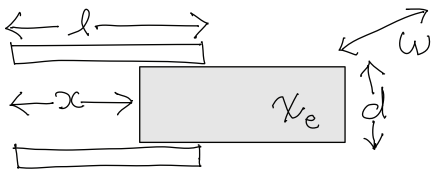
~ due to $\frac{\partial E_z}{\partial x}$ which equals $\frac{\partial E_x}{\partial z}$

$$\vec{F} = -\nabla W = \nabla(\vec{d}\cdot\vec{E}) = \vec{d}\times(\cancel{\nabla\times\vec{E}}) + (\vec{d}\cdot\nabla)\vec{E}$$



* Example: dielectric being pulled into a capacitor

$$C = C_1 + C_2 = \varepsilon_0\frac{xw}{d} + \varepsilon_0(1+\chi_e)\frac{(\ell-x)w}{d} = \frac{\varepsilon_0 w}{d}(\varepsilon_r\ell - \chi_2 x)$$



$$F = -\nabla W = -\frac{d}{dx}\frac{1}{2}CV^2 = -\frac{d}{dx}\frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx}$$

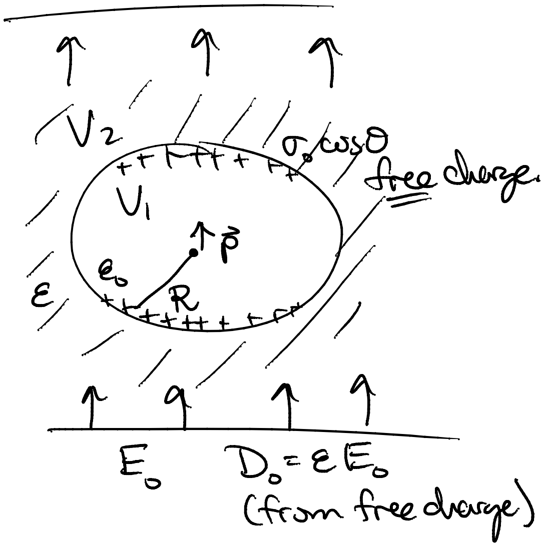
$$= \frac{1}{2}V^2\frac{dC}{dx} = -\frac{\varepsilon_0\omega\chi_e}{2d}V^2$$

~ with constant V, the force would be the same $\quad F = -\nabla W$

but in this case W would increase

Dipole in dielectric & External field & Free charge $\sigma = \sigma_0 \cos\theta$

Friday, December 07, 2012
9:17 AM



$$V = \sum_{\ell=0}^{\infty} (a_\ell r^\ell + b_\ell r^{-\ell-1}) P_\ell(\cos\theta)$$

$$V_{\vec{p}} = \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3} = \underbrace{\frac{\boxed{p}^{\,Q^{(1)}_{int}}}{4\pi\epsilon_0}}_{b_1} \underbrace{r^{-2}\cos\theta}_{r^{-1-1}P_1} \qquad \ell=1$$

$$V_{E_0} = -\int E_0 dz = -E_0 z = \underbrace{\boxed{-E_0}^{\,Q^{1}\cancel{\ell}/4\pi\epsilon_0}}_{a_1} \underbrace{r^1 \cos\theta}_{r^1 P_1} \qquad \ell=1$$

$$V_1 = \frac{p}{4\pi\epsilon_0} r^{-2} P_1 + \sum_{\ell=0}^{\infty} a_\ell r^\ell P_\ell$$

$$V_2 = -E_0 \, r \, P_1 + \sum_{\ell=0}^{\infty} b_\ell r^{-\ell-1} P_\ell$$

* Apply boundary conditions:

$$\Delta V|_\partial = 0: \quad \left(-E_0 R P_1 + \sum_{\ell=0}^{\infty} b_\ell R^{-\ell-1} P_\ell\right) - \left(\frac{p}{4\pi\epsilon_0} R^{-2} P_1 + \sum_{\ell=0}^{\infty} a_\ell R^\ell P_\ell\right) = 0$$

$$-\Delta \varepsilon \frac{\partial V}{\partial n}\Big|_\partial = 0: \quad -\varepsilon_r\left(-E_0 P_1 + \sum_{\ell=0}^{\infty} (-\ell-1) b_\ell R^{-\ell-2} P_\ell\right) + \left(\frac{p}{4\pi\epsilon_0}(-2)R^{-3} P_1 + \sum_{\ell=0}^{\infty} \ell a_\ell R^{\ell-1} P_\ell\right) = \sigma_0/\epsilon_0 \, P_1$$

* Separate out components:

if $\ell \neq 1$: $b_\ell R^\ell - a_\ell R^{-\ell-1} = 0$

$-\varepsilon_r(-\ell-1) b_\ell R^{-\ell-2} + \ell a_\ell R^{\ell-1} = 0$ $\Rightarrow a_\ell = b_\ell = 0$ no "source" term

$\ell = 1$: $-E_0 \cancel{R} + \boxed{b_1 R^{-2\,3}} - \boxed{\frac{p}{4\pi\epsilon_0} R^{-2\,3}} - a_1 \cancel{R} = 0$

$$\varepsilon_r E_0 + 2\varepsilon_r \underbrace{\boxed{b_1 R^3}}_{b'} - 2\underbrace{\boxed{\frac{p}{4\pi\epsilon_0} R^3}}_{p'} + a_1 = \underbrace{\boxed{\sigma_0/\epsilon_0}}_{\sigma'}$$

$b' - a_1 = E_0 + p'$ (1)

$2\varepsilon_r b' + a_1 = -\varepsilon_r E_0 + 2p' + \sigma'$ (2)

source terms on right

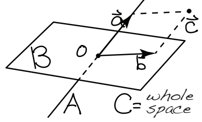
$(2) - 2\varepsilon_r(1)$: $(1 + 2\varepsilon_r)\boxed{a_1} = (-\varepsilon_r - 2\varepsilon_r)E_0 + (2 - 2\varepsilon_r)p' + \sigma'$

$(2) + (1)$: $(1 + 2\varepsilon_r)\boxed{b_1} = (-\varepsilon_r + 1)E_0 + (2+1)p' + \sigma'$

Review of Electrostatics (Chapters 1-4)

* Chapter 1: Mathematics - Vector Calculus

~ Vectors (it's all about being Linear!)
linear combinations; projections
basis (independence, closure)



~ Metric & Cross Product (Bilinear)
orthonormal basis $\hat n\cdot\hat n=\hat n\hat n\cdot-\hat n\times\hat n\times$
longitudinal / transverse projections

$$\vec x=\vec b_i x^i\equiv\sum_{i=1}^{3}\vec b_i x^i$$

$$\hat e_i\cdot\hat e_j=\delta_{ij}$$

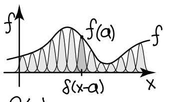
$$\sum_{i=1}^{3}\hat e_i\hat e_i\cdot=I$$

~ Linear Operators
eigenstuff: rotations / stretches

~ Function Spaces - continuous vs discrete
Sturm-Liouville (orthogonal eigenfunctions)

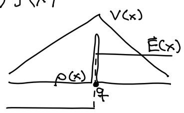
~ Vector Derivatives and Integrals (linearization)
Differentials ordered naturally by dimension

~ Delta function (pole)



$$a_i=\sum_{j=1}^{n}\delta_{ij}a_j \qquad f(x)=\int_{-\infty}^{\infty}dx'\,\delta(x-x')\,f(x')$$

Greens function (tent/pole)



$$\nabla^2G(\vec r)=\nabla^2\frac{-1}{4\pi r}=\nabla\cdot\frac{\hat r}{4\pi r^2}=\delta^3(\vec r)$$

~ Poincare: potentials & derivative chain

$$d\omega=0\iff\omega=d\alpha \qquad \nabla\times\vec E=\vec 0\leftrightarrow\vec E=-\nabla V \qquad \nabla\cdot\vec B=0\leftrightarrow\vec B=\nabla\times\vec A$$

~ Helmholtz theorem: source and potential

$$\vec F=-\nabla\Big(\underbrace{-\nabla^{-2}\overbrace{\nabla\cdot\vec F}^{\rho}}_{V}\Big)+\nabla\times\Big(\underbrace{-\nabla^{-2}\overbrace{\nabla\times\vec F}^{\vec J}}_{\vec A}\Big)$$

$$=-\nabla V+\nabla\times\vec A \qquad \nabla^2(V,\vec A)=-(\rho,\vec J) \qquad \begin{matrix}\nabla\cdot\vec F=\rho\\ \nabla\times\vec F=\vec J\end{matrix}$$

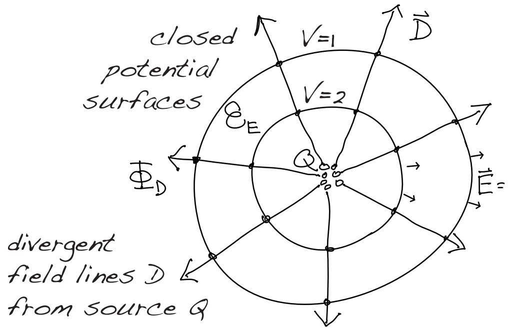
STOKES

$$\oint_{\partial R}\omega=\int_R d\omega$$

RANK		REGION	INTEGRAL	THEOREM	DERIVATIVE	GEOMETRY	
scalar	∂	Q point	$\Delta f=f\big\|_a^b$ change	$f\big\|_a^b=\int_a^b\nabla f\cdot d\vec\ell$ ×	$df=\nabla f\cdot d\vec\ell$ ×	level surface	d
vector	∂	P path	$\mathcal{E}_A=\oint_{\partial S}\vec A\cdot d\vec\ell$ flow	$\oint_{\partial S}\vec A\cdot d\vec\ell=\int_S\nabla\times\vec A\cdot d\vec a$	$d\vec A\cdot d\vec\ell=\nabla\times\vec A\cdot d\vec a$	flow sheets	d
p-vector	∂	S surface	$\Phi_B=\oint_{\partial V}\vec B\cdot d\vec a$ flux	$\oint_{\partial V}\vec B\cdot d\vec a=\int_V\nabla\cdot\vec B\,d\tau$	$d\vec B\cdot d\vec a=\nabla\cdot\vec B\,d\tau$	flux tubes	d
p-scalar	∂	V volume	$Q_\rho=\int_V\rho\,d\tau$ subst.	×	×	subst boxes	d

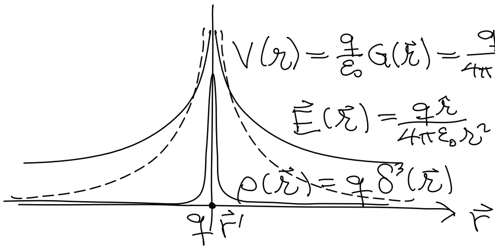
* Chapter 2: Formulations of Electrostatics

Integral	Differential	Boundary
$\vec E=\int\frac{dq'\,\hat r}{4\pi\varepsilon_0 r^2}$	$\nabla^2\vec E=\nabla\rho/\varepsilon_0$	$dq=\sigma da$
$\mathcal{E}_E=0$	$\nabla\times\vec E=0$	$\Delta\hat n\times\vec E=E_{2t}-E_{1t}=0$
$\Phi_D=Q$	$\nabla\cdot\vec D=\rho$	$\Delta\hat n\cdot\vec D=D_{2n}-D_{1n}=\sigma_f$
$V=-\int\vec E\cdot d\vec\ell$	$\vec E=-\nabla V$	$V_2-V_1=0$
$V=\int\frac{dq'}{4\pi\varepsilon_0 r}$	$\nabla^2V=-\rho/\varepsilon_0$	$-\varepsilon_2\partial_nV_2+\varepsilon_1\partial_nV_1=\sigma_f$



Relation between potential, field, and source:

$$\begin{tikzcd}
V \arrow[r, shift left, "-\nabla V"] \arrow[rr, bend right=40, "-\varepsilon_0\nabla^2 V"'] & \vec E \arrow[l, shift left, "-\int\vec E\cdot d\vec\ell"] \arrow[r, shift left, "\nabla\cdot\varepsilon_0\vec E"] \arrow[d, phantom, "\frac{d}{\int}"] & \rho \arrow[l, shift left, "\int\frac{dq'\,\hat r}{4\pi\varepsilon_0 r^2}"] \arrow[ll, bend left=60, "\int\frac{dq'\,\hat r}{4\pi\varepsilon_0 r^2}"] \arrow[d, phantom, "dq'=\rho(\vec r')d\tau'"] \\
& {} & {}
\end{tikzcd}$$



$$V(r)=\frac{q}{\varepsilon_0}G(\vec r)=\frac{q}{4\pi\varepsilon_0 r}$$

$$\vec E(\vec r)=\frac{q\hat r}{4\pi\varepsilon_0 r^2}$$

$$\rho(\vec r)=q\,\delta^3(\vec r)$$

~ Work and Electric field energy: flux x flow

$$\vec F=q\vec E \qquad \vec F=m\vec g$$

$$\underset{\text{potential}=V}{W=q\underbrace{Ed}} \qquad \underset{\text{potential danger}}{W=m\underbrace{gh}}$$

$$\Delta W=\int\Delta\rho_f V\,d\tau=\oint d\vec a\cdot(\Delta\vec D\,V)-\int\Delta\vec D\cdot\vec E\,d\tau$$

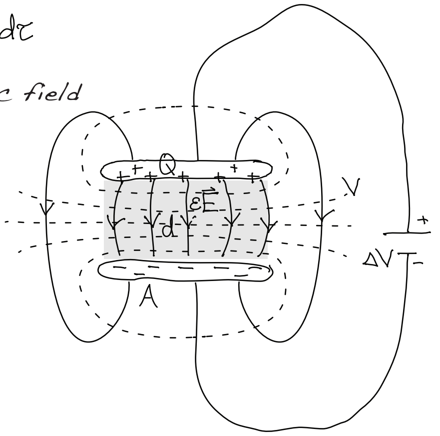
$$W=\int\tfrac{1}{2}\vec D\cdot\vec E\,d\tau$$ energy density of electric field

~ Conductors and Capacitance: flux / flow

ρ=E=0, V=const inside; D=σ, V laminar outside conductor

$$C=\frac{Q}{\Delta V}=\frac{\Phi_D}{\mathcal{E}_E} \qquad Q=\int d\vec a\cdot\vec D=\Phi_D \text{ (closed surface)}$$

$$=\frac{\varepsilon\Phi_E}{\mathcal{E}_E}\approx\frac{\varepsilon A}{d} \qquad \Delta V=\int d\vec\ell\cdot\vec E=\mathcal{E}_E \text{ (open path)}$$



* Chapter 3: Solutions of LaPlace Equation

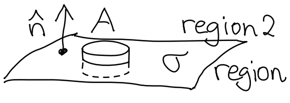
~ Uniqueness Theorem for exterior boundary conditions

$$0 = \int_{\partial V} da\, \underbrace{U}_{(a)}\, \underbrace{\frac{\partial U}{\partial n}}_{(b)} = \oint_{\partial V} d\vec{a}\cdot(U\nabla U) = \int_V \nabla\cdot(U\nabla U)\, d\tau = \int_V U\nabla^2 U + (\nabla U)^2\, d\tau$$

a) Dirichlet B.C. specifies potential on boundary; b) Neuman B.C. specifies flux on boundary

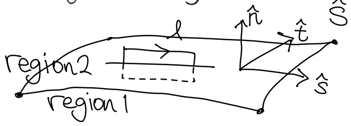
~ continuity boundary conditions stitch potentials together in adjacent regions

Flux: $\vec{D} \equiv \varepsilon\vec{E}$



$\hat{n}\cdot(\vec{D}_2-\vec{D}_1) = \sigma \qquad -\varepsilon_2\frac{\partial V_2}{\partial n}+\varepsilon_1\frac{\partial V_2}{\partial n} = \sigma_f$

Flow:

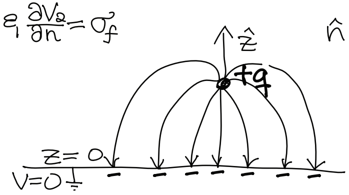


$\hat{S}\times\hat{t}=\hat{n}$

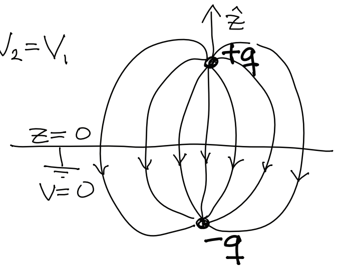
$\hat{n}\times(\vec{E}_2-\vec{E}_1)=0 \qquad V_2=V_1$

A) METHOD OF IMAGES

find a point charge distribution
with the same B.C.'s
same solution by uniques theorem



is equivalent to



B) METHOD OF SEPARATION OF VARIABLES

separate Laplacian (10 known coordinate systems)
solve Sturm-Louiville ODE in each dimension
match boundary conditions to find coefficients
Fourier trick : orthogonal basis functions

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{\vec{k}\cdot\vec{r}} = \sum_{n=1}^{\infty} C_n e^{-k_n x}\sin(k_n y)$$

$$\phi_n(x) = \sin(k_n x) \qquad V(x) = \sum_{n=1}^{\infty} C_n\phi_n(x)$$

$$\langle\phi_n|\phi_m\rangle = \int_0^a \sin(k_n x)\cdot\sin(k_m x)dx = \frac{a}{2}\delta_{nm}$$

$$C_m = \langle\phi_m(x)|V(x)\rangle / \frac{a}{2}$$

C) METHOD OF MULTIPOLE MOMENTS

series expansion of potential about origin or infinity

$$V_1(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^2}\int dq'\, r'\cos\theta' = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3} \qquad \vec{p} = \int dq'\,\vec{r}'$$

$$V(r,\theta) = \sum_{l-0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

$$V_2(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^3}\int dq'\, r'^2\frac{1}{2}(3\cos^2\theta'-1) = \frac{1}{4\pi\varepsilon_0 r^5}\int dq'\,\frac{1}{2}(3(\vec{r}\cdot\vec{r}')^2 - r^2 r'^2)$$

$$= \frac{\frac{1}{2}\vec{r}\cdot\overleftrightarrow{Q}\cdot\vec{r}}{4\pi\varepsilon_0 r^5} \qquad \overleftrightarrow{Q} = \int dq'(3\vec{r}'\vec{r}' - I r'^2) = \int dq'\begin{pmatrix} 3x'^2-r'^2 & 3x'y' & 3x'z' \\ 3y'x' & 3y'^2-r'^2 & 3y'z' \\ 3z'x' & 3z'y' & 3z'^2-r'^2 \end{pmatrix}$$

$$Q_{ij} = \int dq'\,(3r_i'r_j' - \delta_{ij}r'^2)$$

$$Q^{(l)}_{ext} = \frac{A_l}{4\pi\varepsilon_0} = \int dq'\,\frac{1}{r'^{l+1}}P_l(\cos\theta')$$

$$Q^{(l)}_{int} = \frac{B_l}{4\pi\varepsilon_0} = \int dq'\, r'^l\, P_l(\cos\theta')$$

* Chapter 4: Dielectric Materials - Dipole

$\vec{N} = \vec{p}\times\vec{E} \qquad U = -\vec{p}\cdot\vec{E} \qquad \vec{F} = \nabla(\vec{p}\cdot\vec{E})$

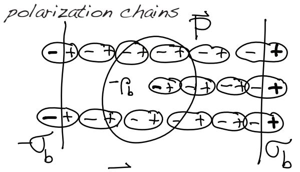
$$\vec{p} = \alpha\vec{E} \qquad \varepsilon_0\chi_e \cong N\alpha$$

$$\vec{P} = \varepsilon_0\chi_e\vec{E} = \frac{\Delta\vec{p}}{\Delta\tau} = \frac{\Delta n}{\Delta\tau}\vec{p} = N\alpha\vec{E}$$

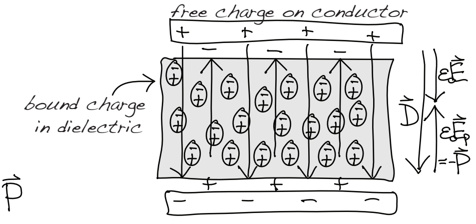
$$\vec{D} = \varepsilon_0\vec{E}+\vec{P} = \varepsilon_0\underbrace{(1+\chi_e)}_{\varepsilon_r}\vec{E} = \underbrace{\varepsilon_0\varepsilon_r}_{\varepsilon}\vec{E} = \varepsilon\vec{E}$$

$$\varepsilon_r = 1+\chi_e = \varepsilon/\varepsilon_0$$

$$\nabla\cdot\vec{D} = \nabla\cdot\varepsilon_0\vec{E} + \nabla\cdot\vec{P} = \rho - \rho_b = \rho_f$$



$\sigma_b = \vec{P}\cdot\hat{n} \qquad \rho_b = -\nabla\cdot\vec{P}$



$D=\varepsilon_0 E+P \qquad 4=1+3 \qquad \varepsilon_r=1+\chi_e$

* Outlook - road to electrodynamic equations

$$\begin{tikzcd}
\lambda \arrow[r, "d"] & (V,\vec{A}) \arrow[r, "d"] & (\vec{E},\vec{B}) \arrow[r, "d"] \arrow[d, equal, "\varepsilon"', "\mu"] \arrow[dr, leftrightarrow, "{\sigma,\vec{F}}"] & 0 & \\
 & (\vec{C},U) \arrow[r, "d"] & (\vec{D},\vec{H}) \arrow[r, "d"] & (\rho,J) \arrow[r, "d"] & 0 \\
(0) & (1) & (2) & (3) & (4)
\end{tikzcd}$$

$$\Phi_D = Q_{encl} \qquad \Phi_B = 0 \qquad -\square^2(V,\vec{A}) = (\rho/\varepsilon, \mu\vec{J})$$

$$\mathcal{E}_E = -\frac{\partial\Phi_B}{\partial t} \qquad \mathcal{E}_H = I_{encl}+\frac{\partial\Phi_D}{\partial t}$$

(wave equation)

$\vec{F} = q(\vec{E}+\vec{v}\times\vec{B}) = (\rho\vec{E}+\vec{J}\times\vec{B})d\tau$ Lorentz force

$\partial_t\rho + \nabla\cdot\vec{J} = 0$ Continuity

$\nabla\cdot\vec{D} = \rho \qquad \nabla\times\vec{E}+\partial_t\vec{B} = \vec{0}$ Maxwell electric,

$\nabla\cdot\vec{B} = 0 \qquad \nabla\times\vec{H}-\partial_t\vec{D} = \vec{J}$ magnetic fields

$\vec{D} = \varepsilon\vec{E} \qquad \vec{J} = \sigma\vec{E} \qquad \vec{B} = \mu\vec{H}$ Constitution

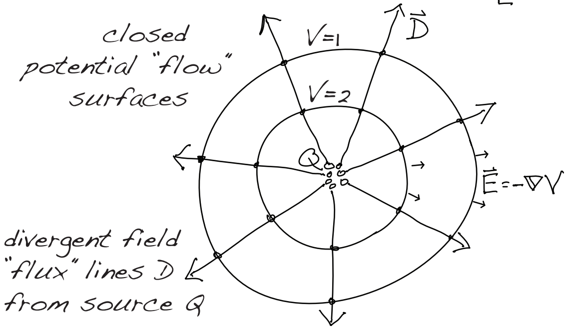
$\vec{E} = -\nabla V - \partial_t\vec{A} \qquad \vec{B} = \nabla\times\vec{A}$ Potentials

$V \to V - \partial_t\lambda \qquad \vec{A} \to \vec{A}+\nabla\lambda$ Gauge transform

Survey of Magnetism and Electrodynamics (Chapters 5-11)

* Electrostatics - Coulomb's law

$$\vec{F}_e = \frac{1}{4\pi\varepsilon_0}\iint dq_1\, dq' \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \int dq_1 \underbrace{\left[\frac{1}{4\pi\varepsilon_0}\int \frac{dq'\,\hat{\mathscr{r}}}{\mathscr{r}^2}\right]}_{\vec{E}}$$

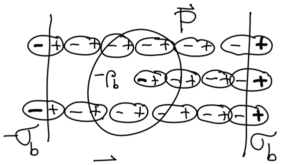


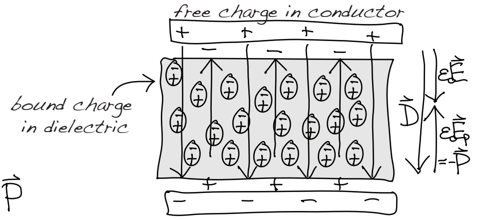
* Helmholtz theorem: source and potential

$$\vec{F} = -\nabla\Big(\underbrace{-\nabla^{-2}\overbrace{\nabla\cdot\vec{F}}^{\rho}}_{V}\Big) + \nabla\times\Big(\underbrace{-\nabla^{-2}\overbrace{\nabla\times\vec{F}}^{\vec{J}}}_{\vec{A}}\Big)$$

$$= -\nabla V + \nabla\times\vec{A} \qquad \nabla^2(V,\vec{A}) = -(\rho,\vec{J}) \qquad \begin{array}{l}\nabla\cdot\vec{F}=\rho\\ \nabla\times\vec{F}=\vec{J}\end{array}$$

* Macroscopic media - polarization chains





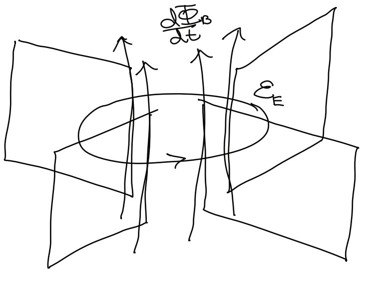
$$\sigma_b = \vec{P}\cdot\hat{n} \qquad \rho_b = -\nabla\cdot\vec{P}$$

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon_0\underbrace{(1+\chi_e)}_{\varepsilon_r}\vec{E} = \underbrace{\varepsilon_0\,\varepsilon_r}_{\varepsilon}\vec{E} = \varepsilon\vec{E}$$

$$\vec{B} = \mu_0(\vec{H}+\vec{M}) = \mu_0\underbrace{(1+\chi_m)}_{\mu_r}\vec{H} = \underbrace{\mu_0\mu_r}_{\mu}\vec{H} = \mu\vec{H}$$

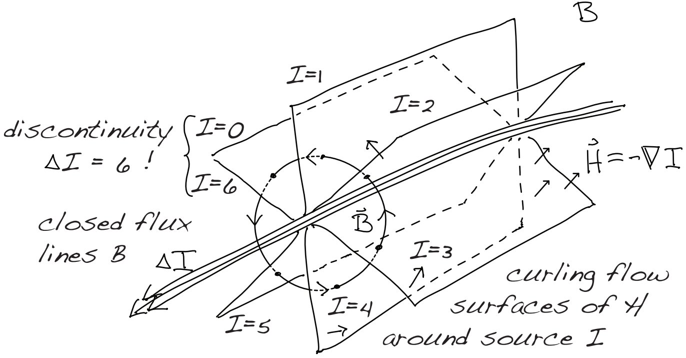
* Faraday's law

$$\mathcal{E}_E = -\frac{\partial\Phi_B}{\partial t}$$



* Magnetostatics - Biot-Savart law

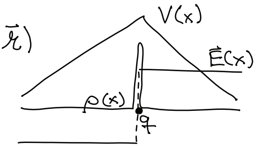
$$\vec{F}_m = \frac{\mu_0}{4\pi}\iint \frac{I\,d\vec{l}\cdot I'\,d\vec{l}'\,\hat{\mathscr{r}}}{\mathscr{r}^2} = \int I\,d\vec{l} \times \underbrace{\left[\frac{\mu_0}{4\pi}\oint\frac{I'\,d\vec{l}'\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right]}_{\vec{B}}$$



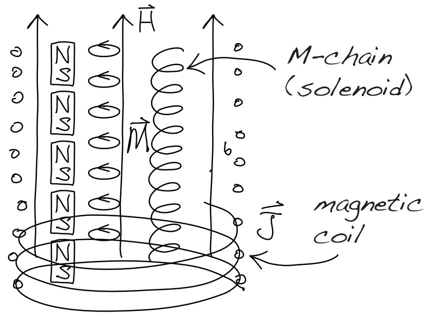
* Green's function (tent/pole)

$$\nabla^2 G(\vec{\mathscr{r}}) = \nabla^2\frac{-1}{4\pi\mathscr{r}} = \nabla\cdot\frac{\hat{\mathscr{r}}}{4\pi\mathscr{r}^2} = \delta^3(\vec{\mathscr{r}})$$

$$G(\vec{\mathscr{r}}) = \frac{-1}{4\pi\mathscr{r}} = \nabla^{-2}\delta^3(\vec{\mathscr{r}})$$

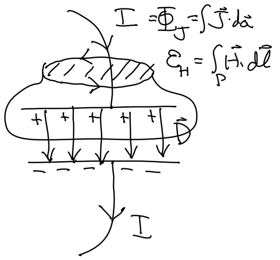


* magnetization chains



* Maxwell's displacement current

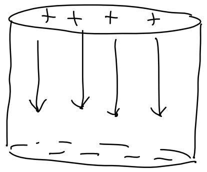
$$\Phi_J = I = \frac{dQ}{dt} = \frac{d\Phi_D}{dt}$$



* Three electrical devices, each the ratio of flux / flow

CAPACITOR

$Q = \int I\,dt$

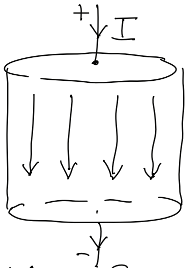


$\vec D = \varepsilon \vec E$

$\nabla \cdot \vec D = \rho$

$C = Q/V = \varepsilon\, \Phi/\mathcal{E} = \varepsilon \frac{A}{l}$

RESISTOR

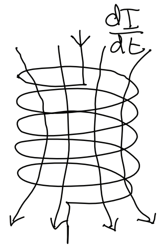


$\vec J = \sigma \vec E$

$\nabla \cdot J = \frac{d\rho}{dt}$

$R = V/I = \mathcal{E}/\sigma\Phi = \frac{l}{\sigma A}$

INDUCTOR



$\vec B = \mu \vec H$

$\nabla \times \vec H = \vec J$

$L = V/\dot I = \Phi/I = N^2 \mu\, \Phi/\mathcal{E} = N^2 \frac{\mu A}{l}$

* Electrodynamics equations

$$\begin{tikzcd}
\lambda \arrow[r, "d"] & (V,\vec A) \arrow[r, "d"] & (\vec E,\vec B) \arrow[r, "d"] \arrow[d, Rightarrow, "\varepsilon"', "\mu"] \arrow[dr, leftrightarrow, "{\sigma, \vec F}"] & 0 & \\
 & (\vec C, I) \arrow[r, "d"] & (\vec D,\vec H) \arrow[r, "d"] & (\rho, J) \arrow[r, "d"] & 0 \\
(0) & (1) & (2) & (3) & (4)
\end{tikzcd}$$

$\Phi_D = Q_{encl}$ $\quad \Phi_B = 0$

$\mathcal{E}_E = -\frac{\partial \Phi_B}{\partial t}$ $\quad \mathcal{E}_H = I_{encl} + \frac{\partial \Phi_D}{\partial t}$

$-\Box^2(V,\vec A) = (\rho/\varepsilon, \mu\vec J)$

(wave equation)

$\vec F = q(\vec E + \vec v \times \vec B) = (\rho \vec E + \vec J \times \vec B)d\tau$		Lorentz force
$\partial_t \rho + \nabla \cdot \vec J = 0$		Continuity
$\nabla \cdot \vec D = \rho$	$\nabla \times \vec E + \partial_t \vec B = \vec 0$	Maxwell electric,
$\nabla \cdot \vec B = 0$	$\nabla \times \vec H - \partial_t \vec D = \vec J$	magnetic fields
$\vec D = \varepsilon \vec E$ $\quad \vec J = \sigma \vec E$	$\vec B = \mu \vec H$	Constitution
$\vec E = -\nabla V - \partial_t \vec A$	$\vec B = \nabla \times \vec A$	Potentials
$V \to V - \partial_t \lambda$	$\vec A \to \vec A + \nabla \lambda$	Gauge transform

* Conserved currents

$$T_{\mu\nu} = \begin{array}{cc} \text{desity} & \text{flux} \\ \left(\begin{array}{cc} u & \vec S \\ \vec{\wp} & \overleftrightarrow{T} \end{array}\right) \end{array} \begin{array}{l} \text{energy} \\ \text{momentum} \end{array}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec J = \partial_\mu J^\mu = 0$$

$$\frac{\partial}{\partial t}(u_{mech} + u_{em}) + \nabla \cdot \vec S = 0 \qquad \vec S \equiv \vec E \times \vec H$$

$$\frac{\partial \wp}{\partial t} + \nabla \cdot \overleftrightarrow{T} = 0 \qquad \overleftrightarrow{T} \equiv (\vec D \vec E + \vec B \vec H) - \frac{1}{2}(\vec D \cdot \vec E + \vec B \cdot \vec H)\, I$$

* Electromagnetic waves

- Fresnell's coefficients
- skin depth
- dipole radiation

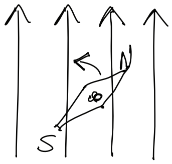
Section 5.1.1 - Magnetic Fields

- the magnetic force was known in antiquity, but was more difficult to quantify
  - ~ predominant effect in nature involves magnetization, not electric currents
  - ~ no magnetic (point) charge (monopole); 1-d currents instead of 0-d charges
  - ~ static electricity was produced in the lab long before steady currents

- History: from "A Ridiculous Brief History of Electricity and Magnetism"

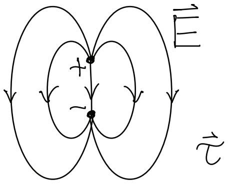
600 BC	Thales of Miletus discovers lodestone's attraction to iron
1200 AD	Chinese use lodestone compass for navigation
1259 AD	Petrus Peregrinus (Italy) discovers the same thing
1600 AD	William Gilbert discovers that the Earth is a giant magnet
1742 AD	Thomas LeSeur shows inverse cube law for magnets
1820 AD	Hans Christian Oersted discovers that current twists magnets
1820 AD	Andre Marie Ampere shows that parallel currents attract/repel
1820 AD	Jean-Baptiste Biot & Felix Savart show inverse square law

- for magnetism it is much more natural to start with the concept of field



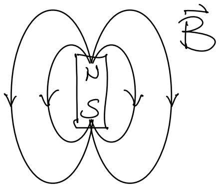
~ compass points N because it aligns with the Earth's magnetic field

~ iron filings chain up to show physical "field lines"



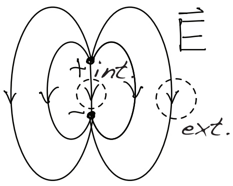
~ bar magnet field lines resemble an electric dipole

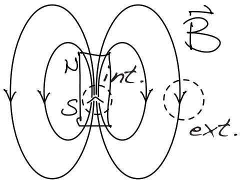
$\vec{\tau} = \vec{p} \times \vec{E} \qquad \vec{\tau} = \vec{m} \times \vec{B}$



- what is the main difference?
  - ~ two differences related to "flux" and "flow"
  - ~ difference between "internal" and "external" dipole

Amber (electric)	Lodestone (magnet)
~ rub to charge	~ always charged
~ direct force	~ torque
~ 2 charges +/-	~ 2 poles (N/S)
~ fluid monopoles	~ unseparable dipole



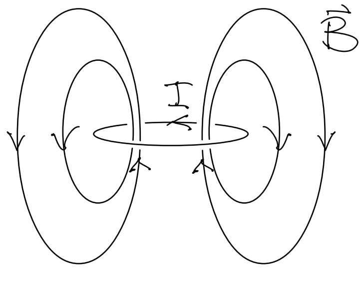


~ sources/sinks of flux

~ conservative flow (potential)

~ conserved flux lines (solenoidal)

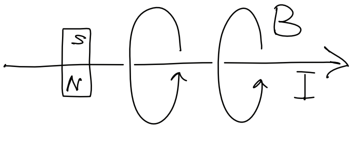
~ source of flow (rotational)



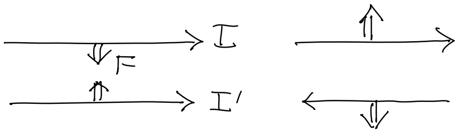
- no magnetic monopole!
  - ~ N/S poles cannot be separated
  - ~ reason: magnetic dipoles are actually current loops
  - ~ note: field lines are perpendicuar to source current

~ this is the source of the differences between E vs. B eq. and dielecrics vs. magnets

- discovery by Hans Christian Oersted (1820)
  - ~ current produces a magnetic field
  - ~ generalized to the force between wires by Ampere, Biot and Savart



- Ampere $\vec{F}/\ell = \left(\frac{\mu_0}{2\pi}\frac{I'}{d}\right) I = \vec{I} \times \vec{B}$



~ for two wires separated by distance d

~ definition of Ampere [A], Tesla [T], Gauss [G] (CGS units), Coulomb [C]

$$\frac{\mu_0}{4\pi} \equiv 10^{-7}\, N/A^2 = 1\, \frac{G\ mm}{A} \qquad 1\,T = 1\, N/A\cdot m \doteq 10^4\, G \qquad 1\,C = 1\,A\cdot s$$

note different dimensions: $\mu_0 \equiv 1$ [CGS]

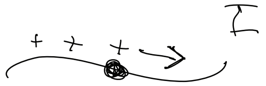
Section 5.1.2 - Lorentz Force, Current Elements

* magnetic force law $\vec{F} = BIl = I\int d\vec{l}\times\vec{B} = q\,\vec{v}\times\vec{B}$

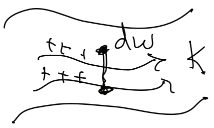
~ the combination $I d\vec{l}$ occurs frequently, it is called the "current element"

~ units: A m = C m/s ~ qv, much like a "charge element" $dq = \lambda dl = \sigma da = \rho d\tau$

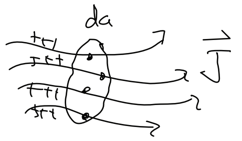
* current density $dq = I\,dt = K\,dw\,dt = J\,da\,dt$



$$I = \frac{dq}{dt} = \lambda\vec{v}$$



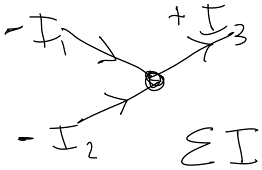
$$K = \frac{dq}{dw\,dt} = \sigma\vec{v}$$



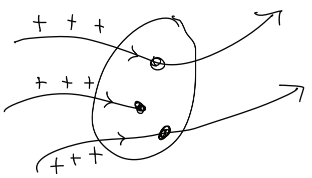
$$J = \frac{dq}{da\,dt} = \frac{dq}{d\tau}\,\frac{dl}{dt} = \rho\vec{v}$$

* conservation of charge: Kirchov's current law

$$I_{tot} = \Phi_J = -\frac{dQ}{dt}$$



$$\Sigma I = -\frac{dq}{dt}$$



$$\int_V \nabla\cdot\vec{J}\,d\tau = \oint_{\partial V}\vec{J}\cdot d\vec{a} = \int_V \rho\,d\tau$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0$$

~ written as 4-vectors

$$\left(\frac{\partial}{\partial t}\ \frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z}\right)\begin{pmatrix}\rho\\ J_x\\ J_y\\ J_z\end{pmatrix} = 0.$$

* relation between charge and current elements

$$I d\vec{l} \sim \vec{K} da \sim \vec{J} d\tau$$

$$dq\,\vec{v} \sim \lambda\vec{v}\,dl \sim \sigma\vec{v}\,da \sim \rho\vec{v}\,d\tau$$

* Lorentz force law $\boxed{d\vec{F} = dq\,(\vec{E} + \vec{v}\times\vec{B})}$

dq is any "charge element"

$dq\,\vec{v}$ is any "current element"

* magnetic forces do no work

$\vec{E}$ tangential acceleration (not quite) "gas pedal"

$\vec{B}$ radial acceleration (always) "steering wheel"

$$dW_m = \vec{F}_m\cdot d\vec{l} = Q(\vec{v}\times\vec{B})\cdot\vec{v}\,dt = 0$$

~ similar to the normal force which only deflects objects

Sections 5.1.3, 8.1.1 - Conserved currents: continuity eq.

* Symmetries: $\frac{d}{dt}\underbrace{\frac{\partial T}{\partial \vec{v}}}_{\vec{p}} - \underbrace{\frac{-\partial V}{\partial \vec{x}}}_{\vec{F}} = 0$ (Lagrange)

$$\frac{\partial T}{\partial \vec{v}} = \frac{d}{d\vec{v}}\frac{1}{2}m\vec{v}^2 = m\vec{v} = \vec{p}$$

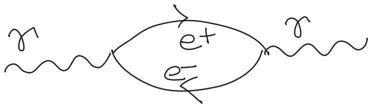
$$\frac{d\vec{p}}{dt} = m\vec{a} = \vec{F} = -\frac{\partial V}{\partial \vec{x}}$$

N III: $F_{21} = -F_{12}$

~ if $\mathcal{L}$ is translation invariant (symm'metric)
then momentum ($\vec{p}$) is conserved in complete system

~ if laws of physics (forces) are time-invariant
then energy (E) is conserved (potential energy is stored in the force)

* Noether's thoorem: SYMMETRIES $\Longleftrightarrow$ CONSERVED CURRENTS
~ mass?
~ charge?

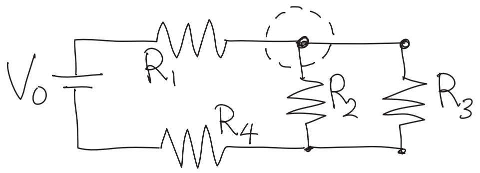


(the quantity is conserved,
but it can move around)

Gauge transformations $\vec{E} = -\nabla V$ (force independent of ground potential)

* Kirchoff's rules: conservation principles
a) loop rule: conservation of energy

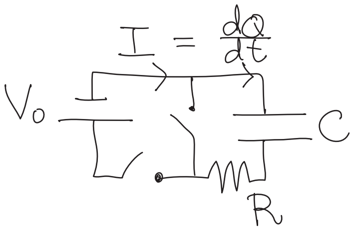
$$\sum_{loop}\Delta V_i = -\oint_{\partial S}\vec{E}\cdot d\vec{l} = -\mathcal{E}_E = -\int_S \nabla\times\vec{E}\cdot d\vec{a} = 0$$



b) node rule: conservation of charge

$$\sum_{node} I_i = 0 = \oint_{\partial V}\vec{J}\cdot d\vec{a} = \int_V \nabla\cdot J\, d\tau$$

~ what about a capacitor?
top plate has current coming in
but no current going out



$V_2 = V_3$

$I_1 = I_2 + I_3$

* charge element vs. $dq = q_i = \lambda\, dl = \sigma\, da = \rho\, d\tau$

current element: $d\vec{q} = \vec{v}_i q_i = I\, d\vec{l} = \vec{K}\, da = \vec{J}\, d\tau$

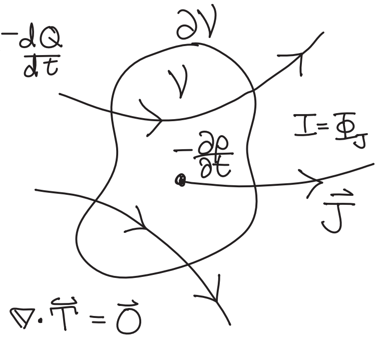
$\vec{I} = \vec{v}\lambda = \frac{\Delta q}{\Delta t}\hat{l}$ $\quad\vec{K} = \vec{v}\sigma = \frac{\Delta q}{\Delta w\Delta t}\hat{l}$ $\quad\vec{J} = \vec{v}\rho = \frac{\Delta q}{\Delta a\Delta t}\hat{l}$

$I = \int \vec{K}\times d\vec{w}$ $\quad I = \int \vec{J}\cdot d\vec{a}$

* continuity equation:
local conservation of charge
vs. "beam me up, Scotty"
~ 4-vector: $(c\rho, \vec{J}) = J^\mu$

$$I = \oint_{\partial V}\vec{J}\cdot d\vec{a} = \int_V \nabla\cdot\vec{J}\, d\tau = \int_V \frac{\partial\rho}{\partial t}\, d\tau = -\frac{dQ}{dt}$$

$$\boxed{\nabla\cdot J = -\frac{\partial\rho}{\partial t}}$$



$$\frac{1}{c}\frac{\partial}{\partial t}c\rho + \nabla\cdot\vec{J} \equiv \vec{\Box}\cdot\vec{J} = 0 \qquad \partial_\mu J^\mu = 0$$

~ other conserved currents: energy $\frac{\partial u}{\partial t} + \nabla\cdot\vec{S} = 0$, momentum $\frac{\partial\vec{p}}{\partial t} + \nabla\cdot\overleftrightarrow{T} = \vec{0}$

Section 5.2 - Biot-Savart Law

* review:
- charge element (scalar): $dq \sim \lambda dl \sim \sigma da \sim \rho d\tau$
- current element (vector!): surface, volume current density $dq\,\vec{v} \sim I d\vec{l} \sim \vec{K} da \sim \vec{J} d\tau$ $\Big\}\times\vec{v}$

$$I = \Phi_J = \int \vec{J}\cdot d\vec{a}$$

- steady currents: analog of electrostatic stationary charges $\Delta I = \frac{dQ}{dt} = 0 \qquad \nabla\cdot\vec{J} = -\frac{\partial\rho}{\partial t} = 0$

* electrostatic vs. magnetostatic force laws
- definition of "force" fields E, B (vs. "source" fields D, H, see next chapter)
- fields mediate force from one charge (current) to another (action at a distance)
- experiment by Oersted defined direction of field, Ampere defined magnitude

~ Coulomb Law (electric)

$$\vec{F}_e = \frac{1}{4\pi\varepsilon_0}\int_{\vec r}\int_{\vec r'} dq \underbrace{dq' \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}} \equiv \int dq\, \vec{E} \qquad \vec{E} \equiv \frac{1}{4\pi\varepsilon_0}\int_{\vec r'} \frac{dq'\hat{\mathscr{r}}}{\mathscr{r}^2} = -\nabla \underbrace{\frac{1}{4\pi\varepsilon_0}\int_{\vec r'}\frac{dq}{\mathscr{r}}} = -\nabla V$$

~ Biot-Savart Law (magnetic)

$$\vec{F}_m = \frac{-\mu_0}{4\pi}\oint_{\vec r}\oint_{\vec r'} I d\vec{l} \cdot \underbrace{I' d\vec{l}' \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}} = \oint I' d\vec{l} \times \vec{B} \qquad \vec{B} \equiv \frac{\mu_0}{4\pi}\oint_{\vec r'} \frac{I' d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2} = ?$$

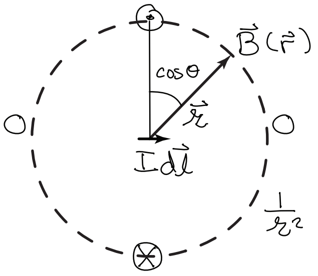
~ proof: $A \times (B \times C) = B\,(A\cdot C) - C\,(A\cdot B)$

$$\vec{F}_m = \oint I d\vec{l} \times \oint' \frac{\mu_0}{4\pi} I' d\vec{l}' \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0}{4\pi}\oint\oint' I' d\vec{l}' \underbrace{\left(I d\vec{l} \cdot -\nabla\frac{1}{\mathscr{r}}\right)}_{\oint I\, d\frac{1}{\mathscr{r}} = 0} - \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\left(I d\vec{l} \cdot I' d\vec{l}'\right)$$

~ combined: Lorentz force law

$$\vec{F} = \int dq\,(\vec{E} + \vec{v}\times\vec{B}) = \int d\tau\,(\rho\vec{E} + \vec{J}\times\vec{B})$$

directional
inverse-square
force law

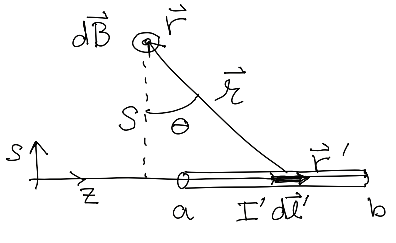


* Example 5.5: Parallel wires

$$\vec{B} = \frac{\mu_0}{4\pi}\int I' d\vec{l}' \times \frac{\vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{\mu_0 I'}{4\pi}\int_{z'=a}^{b} \frac{dz'\,\hat{z} \times (s\,\hat{s} + (z-z')\,\hat{z})}{(s^2 + (z-z')^2)^{3/2}}$$

$$= \frac{\mu_0 I'}{4\pi}\int_{z'=a}^{b} \frac{s\, dz'\,\hat{\phi}}{(s^2 + (z-z')^2)^{3/2}} = \frac{\mu_0 I'}{4\pi}\int \frac{s^2 \sec^2\theta\, d\theta\,\hat{\phi}}{\underbrace{(s^2(1+\tan^2\theta))^{3/2}}_{s^3\sec^3\theta}}$$

$$= \frac{\mu_0 I'}{4\pi s}\int_{z'=a}^{b} \cos\theta\, d\theta\,\hat{\phi} = \frac{\mu_0 I'}{4\pi s}(\sin\theta_b - \sin\theta_a)\,\hat{\phi}$$



let $z - z' = -\tan\theta$

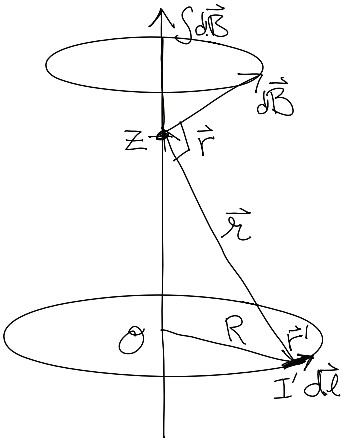
$dz' = s\,\sec^2\theta\, d\theta$

~ for an infinite wire: $\vec{B} = \frac{\mu_0 I'}{2\pi s}\,\hat{\phi}$

~ for a second parallel wire: $\vec{F} = \int_{\vec r} I d\vec{l} \times \vec{B} = -\frac{\mu_0}{2\pi}\frac{I I'}{s}\,\hat{s}\; l$

as shown before, this was unsed to define the Ampere (current) -> Tesla (B-field)

* Example 5.6: Current loop:



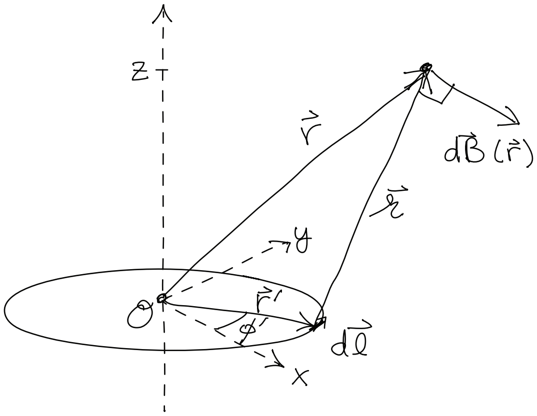
$$\vec{B} = \frac{\mu_0}{4\pi}\oint I' d\vec{l}' \times \frac{\vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{\mu_0}{4\pi}\int_{\phi'=0}^{2\pi} I' R\, d\phi' \hat{\phi} \times \frac{z\hat{z} - R\hat{s}'}{\mathscr{r}^3}$$

$$= \frac{\mu_0}{4\pi}\int_{\phi'=0}^{2\pi} I' R\, d\phi' \frac{z\hat{s}' + R\hat{z}}{\mathscr{r}^3}$$

$$= \frac{\mu_0}{4\pi}\int_{\phi'=0}^{2\pi} I' R\, d\phi' \frac{z(\cos\phi'\hat{x} + \sin\phi'\hat{y}) + R\hat{z}}{\mathscr{r}^3}$$

$$= \frac{\mu_0 I'}{4\pi} 2\pi \frac{R^2\hat{z}}{\mathscr{r}^3} = \frac{\mu_0 I' R^2 \hat{z}}{2(R^2+z^2)^{3/2}}$$

* Example: Off-axis field of current loop:



$$\vec{B} = \frac{\mu_0}{4\pi}\oint I' d\vec{l}' \frac{\vec{\mathscr{r}}}{\mathscr{r}^3}$$

$$= \frac{\mu_0}{4\pi}\int_{\phi'=0}^{2\pi} I' R\, d\phi' \hat{\phi}' \times \frac{z\hat{z} + (s\hat{s} - R\hat{s}')}{(z^2 + (s\hat{s} - R\hat{s}')^2)^{3/2}}$$

$$= \frac{\mu_0 I'}{4\pi}\int_{\phi=0}^{2\pi} R\, d\phi' \hat{\phi}' \times \frac{z\hat{z} + s\hat{x} - R\hat{s}'}{(r^2 + R^2 - 2sR\cos\phi')^{3/2}}$$

$$= \frac{\mu_0 I'}{4\pi}\int_{\phi'=0}^{2\pi} R\, d\phi' (-\cancel{\sin}\phi'\hat{x} + \cos\phi'\hat{y}) \times \frac{(z\hat{z} + s\hat{x} + R\cancel{\sin}\phi'\hat{x} - R\cancel{\cos}\phi'\hat{y})}{(r^2 + R^2 - 2sR\cos\phi')^{3/2}}$$

$$= \frac{\mu_0 I' R}{4\pi}\int_{\phi'=0}^{2\pi} \frac{z\hat{x} - (s+R)\hat{z}\cos\phi'\, d\phi'}{(r^2 + R^2 - 2sR\cos\phi')^{3/2}}$$

Section 5.3 - Div and Curl of B

* the formalism of both electrostatics and magnetostatics follow the Helmholtz theorem
* these two diagrams illustrate the symmetry between the two forces

$$\begin{tikzcd}
\text{Coulomb \& Superposition} & \\
\boxed{\vec{\boldsymbol{E}} = \int \frac{\text{đ}q'\,\hat{\mathscr{r}}}{4\pi\epsilon_0\,\mathscr{r}^2}} \arrow[d] \arrow[dr] & \begin{array}{c}\vec{\boldsymbol{F}} = q\vec{\boldsymbol{E}}\\ W = qV\end{array} \\
\text{Integral field} & \text{Differential field} \arrow[ul, "\text{Helmholtz}"'] \\
\boxed{\begin{array}{l}\Phi_E = Q/\epsilon_0\\ \mathcal{E}_E = 0 \quad \text{(closed regions)}\end{array}} \arrow[r, leftrightarrow, "\text{Gauss}", "\text{Stokes}"'] \arrow[d, leftrightarrow] & \boxed{\begin{array}{l}\nabla\cdot\vec{\boldsymbol{E}} = \rho/\epsilon_0\\ \nabla\times\vec{\boldsymbol{E}} = \boldsymbol{0}\end{array}} \arrow[d, leftrightarrow, "\text{Poincaré}"] \\
\mathcal{E}_E = -\Delta V \arrow[r, leftrightarrow, "\text{FTVC}"] & \vec{\boldsymbol{E}} = -\nabla V \\
\text{Potential} & \text{Poisson} \\
\boxed{V = \int \frac{\text{đ}q'}{4\pi\epsilon_0\,\mathscr{r}}} \arrow[r, leftrightarrow, "\text{Laplace}", "\text{Green}"'] & \boxed{\nabla^2 V = -\rho/\epsilon_0}
\end{tikzcd}$$

$$\begin{tikzcd}
\text{Biot-Savart \& Superposition} & \\
\boxed{\vec{\boldsymbol{B}} = \int \frac{\mu 0}{4\pi}\frac{\vec{\boldsymbol{J}}\,d\tau\times\hat{\mathscr{r}}}{\mathscr{r}^2}} \arrow[d] \arrow[dr] & \begin{array}{c}\vec{\boldsymbol{F}} = q\vec{\boldsymbol{v}}\times\vec{\boldsymbol{B}}\\ \vec{\boldsymbol{p}} = q\vec{\boldsymbol{A}}\end{array} \\
\text{Integral field} & \text{Differential field} \arrow[ul, "\text{Helmholtz}"'] \\
\boxed{\begin{array}{l}\mathcal{E}_B = \mu_0 I\\ \Phi_B = 0 \quad \text{(closed regions)}\end{array}} \arrow[r, leftrightarrow, "\text{Stokes}", "\text{Gauss}"'] \arrow[d, leftrightarrow, "?"] & \boxed{\begin{array}{l}\nabla\times\vec{\boldsymbol{B}} = \mu_0\vec{\boldsymbol{J}}\\ \nabla\cdot\vec{\boldsymbol{B}} = 0\end{array}} \arrow[d, leftrightarrow, "\text{Poincaré}"] \\
{} \arrow[r, leftrightarrow, "?"] & \vec{\boldsymbol{B}} = \nabla\times\vec{\boldsymbol{A}} \\
\text{Potential} & \text{Poisson} \\
\boxed{\vec{\boldsymbol{A}} = \int \frac{\mu_0}{4\pi}\frac{\vec{\boldsymbol{J}}\,d\tau}{\mathscr{r}}} \arrow[r, leftrightarrow, "\text{Laplace}", "\text{Green}"'] & \boxed{\nabla^2\vec{\boldsymbol{A}} = -\mu_0\vec{\boldsymbol{J}}}
\end{tikzcd}$$

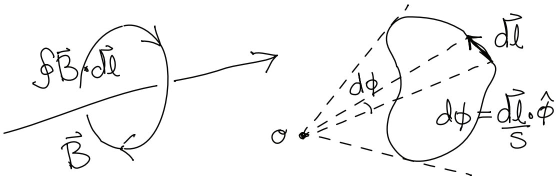
* derivative chains

$$\begin{tikzcd}
V \arrow[r, "-\nabla"] & \vec{E} \arrow[r, "\nabla\times"] \arrow[d, "\varepsilon"'] & 0 \\
& \vec{D} \arrow[r, "\nabla\cdot"] & \rho
\end{tikzcd}
\qquad
\begin{tikzcd}
& & 0 & & \\
\chi \arrow[r, "\nabla"] & \vec{A} \arrow[r, "\nabla\times"] \arrow[ur, bend left, "\nabla\cdot"] & \vec{B} \arrow[r, "\nabla\cdot"] & 0 & \\
& U \arrow[r, "-\nabla"] & \vec{H} \arrow[r, "\nabla\times"] \arrow[u, "\mu"'] & \vec{J} \arrow[r, "\nabla\cdot"] & 0
\end{tikzcd}$$

* integral equations

$$\mathcal{E}_B = \oint \vec{B}\cdot d\vec{l} = \mu_0 I \oint \frac{s\,d\phi}{2\pi s} = \mu_0 I \int_0^{2\pi} \frac{d\phi}{2\pi} = \mu_0 I$$

~ assumes exactly 1 winding, otherwise, $\mu_0 NI$



* differential equations (note: $\text{đ}\vec{q}\,' = \vec{J}'d\tau'$ or $\vec{K}'da'$ or $I'd\vec{l}'$)

$$\begin{aligned}
\vec{B} &= \int' \frac{\mu_0}{4\pi}\,\text{đ}\vec{q}\,' \times \nabla \frac{-1}{\mathscr{r}} \\
&= \nabla\times\int' \frac{\mu_0}{4\pi}\frac{\text{đ}\vec{q}\,'}{\mathscr{r}} \\
&= \nabla\times\vec{A} \qquad \vec{A} = \int' \frac{\mu_0\,\text{đ}\vec{q}\,'}{4\pi\mathscr{r}}
\end{aligned}$$

$$\begin{aligned}
\nabla\cdot\vec{A} &= \nabla\cdot\int' \frac{\mu_0\,\text{đ}\vec{q}\,'}{4\pi\mathscr{r}} \\
&= \frac{\mu_0}{4\pi}\int' \text{đ}\vec{q}\,'\cdot\nabla\frac{1}{4\pi\mathscr{r}} \\
&= \frac{\mu_0}{4\pi}\int' d\tau'\,\vec{J}'\cdot\nabla'\underbrace{\frac{-1}{4\pi\mathscr{r}}}_{G(\mathscr{r})} = 0
\end{aligned}$$

$$\nabla\cdot\vec{B} = \nabla\cdot\nabla\times\vec{A} = 0 \quad \Longleftrightarrow$$

$$\Phi_B = \oint_{\partial S}\vec{B}\cdot d\vec{a} = \int_S \nabla\cdot\vec{B}\,d\tau = 0$$

$$\begin{aligned}
\int'\nabla'\cdot(\vec{J}'G)\,d\tau' &= \int(\cancelto{0}{\nabla'\cdot\vec{J}'})\,G\,d\tau' + \int'\vec{J}'\cdot\nabla'G\,d\tau \\
&= \oint d\vec{a}'\cdot(\vec{J}'G) = 0
\end{aligned}$$

$$\nabla\times\vec{B} = \nabla\times(\nabla\times\vec{A}) = \nabla(\cancelto{0}{\nabla\cdot\vec{A}}) - \nabla^2\vec{A} = \mu_0\vec{J}$$

$$\begin{aligned}
-\nabla^2\vec{A} &= -\nabla^2\int' \frac{\mu_0\,\text{đ}\vec{q}\,'}{4\pi\mathscr{r}} = \int'\mu_0\,\text{đ}\vec{q}\,'\,\nabla^2\underbrace{\frac{-1}{4\pi\mathscr{r}}}_{G(\vec{\mathscr{r}})} \\
&= \int'\mu_0\vec{J}(\vec{r}\,')\,\delta^3(\vec{r}-\vec{r}\,')\,d\tau' = \mu_0\vec{J}(\vec{r})
\end{aligned}$$

$$\mathcal{E}_B = \oint_{\partial S}\vec{B}\cdot d\vec{l} = \int_S \nabla\times\vec{B}\cdot d\vec{a} = \mu_0\int\vec{J}\cdot d\vec{a} = \mu_0 I_{enc}$$

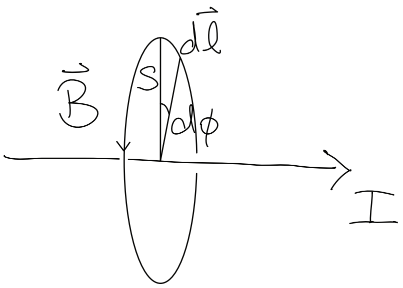
note: $\nabla f(\mathscr{r}) = -\nabla' f(\mathscr{r})$

i.e. $\frac{\partial}{\partial x}f(x-x') = f'\cdot\frac{\partial(x-x')}{\partial x} = +f'$

$\frac{\partial}{\partial x'}f(x-x') = f'\cdot\frac{\partial(x-x')}{\partial x'} = -f'$

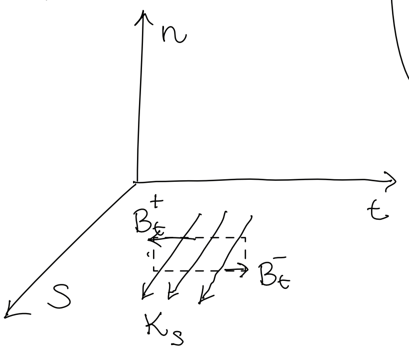
Applications of Ampere's law

* Ampere's law is the analog of Gauss' law for magnetic fields
  ~ uses a path integral around closed loop   instead of integral over a closed surface
  ~ simplest way to solve magnetic fields with high symmetry



* Example 5.7: straight wire

$$\oint \vec{B}\cdot d\vec{l} = B_\phi \cdot 2\pi s = \mu_0 I \qquad \vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$



* Example 5.8: current septum

$$\oint \vec{B}\cdot d\vec{l} = (-B_t^+ + B_t^-)\, l = \mu_0 K_s l = \mu_0 I$$

$$\hat{n} \times \Delta\vec{B}_t = \vec{K} \quad \text{ie.} \quad B_t^\pm = \pm\tfrac{1}{2}K_s\hat{t}$$

* Example 5.9: infinite solenoid
  ~ winding density n = #turns / length

$$K = N\frac{I}{l} = In$$

$$\oint \vec{B}\cdot d\vec{l} = (B_z - B_z)\cdot L = 0 \quad \text{outside.}$$

$$\oint \vec{B}\cdot d\vec{l} = (B_z - B_z)\cdot L = \mu_0 K L \qquad \Delta B = \mu_0 K \text{ again!}$$

* Maxwell's equations (steady-state E&M)

$$\nabla\cdot\vec{E} = \rho/\varepsilon_0 \qquad \nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{E} = 0 \qquad \nabla\times\vec{B} = \mu_0\vec{J}$$

$$\vec{F} = Q(\vec{E} + \vec{v}\times\vec{B})$$

  ~ the two zeros mean there is no magnetic monopole
  ~ actually as long as q_e/q_m is constant, a magnetic monopole can
   turned into an electric charge by a redefinition of E amd B (duality rotation)

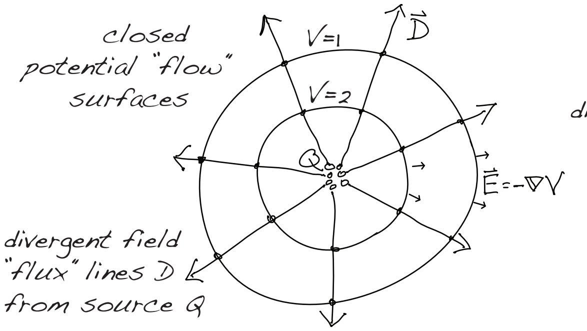
Section 5.x - Magnetic Scalar Potential

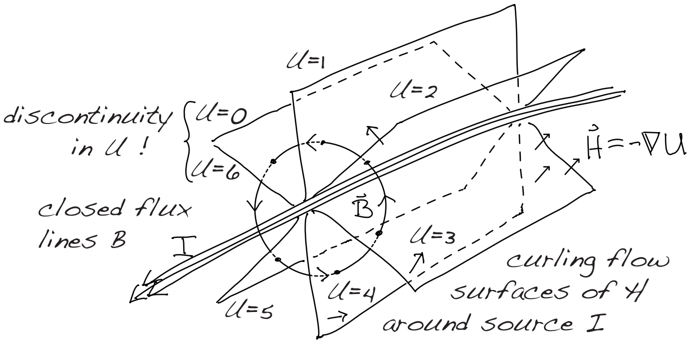
* pictorial representation of Maxwell's steady state equations

~ define $B=\mu_0 H$ to drop all the μ_0's and emphasize the "source" aspect of B

	electric	magnetic	electric	magnetic
flux:	$\nabla\cdot\vec{D}=\rho$	$\nabla\cdot\vec{B}=0$	$\Phi_D=Q$	$\Phi_B=0$
flow:	$\nabla\times\vec{E}=0$	$\nabla\times\vec{H}=\vec{J}$	$\mathcal{E}_E=0$	$\mathcal{E}_H=I$

$$U=\mathcal{E}_H\equiv-\int_0^{\vec{r}}\vec{H}\cdot d\vec{l}$$
$$\vec{H}=-\nabla U$$





* utility of treating D,B as flux lines
and E,H as equipotential surfaces:

~ flux through a surface S: $\Phi_B=\int_S\vec{B}\cdot d\vec{a}$ = # of lines that poke through a surface S

~ flow along a curve/path P: $\mathcal{E}_E=\int_P E\cdot d\vec{l}$ = # of surfaces that a path P pokes through

* scalar electric and magnetic potentials

$\vec{E}=-\nabla V$ ALWAYS $\nabla\times\vec{E}=0$

$\nabla\cdot\varepsilon(-\nabla V)=\rho$ Poisson's eq.

$V=\frac{-1}{\varepsilon}\nabla^{-2}\rho=\frac{-1}{\varepsilon}\int\frac{d\tau'\rho}{r}+\nabla^{-2}0$

$\vec{H}=-\nabla U$ ONLY if if $\vec{J}=0$

$\nabla\cdot\mu(-\nabla U)=0$ Laplace's eq. ALWAYS

~ solve $\nabla^2 U=0$ with appropriate B.C.'s

* discontinuities in U

a) at I: the edge of each H sheet is an I line

b) around I: the U=0,6 surfaces coincide - a 'branch cut' on U extends from each I line

~ U is well defined in a simply connected region or one that does not link any current

* boundary conditions

$\nabla\to\Delta\hat{n}$ $\rho\to\sigma$, $\vec{J}\to\vec{K}$

$$\mathcal{E}_H=\oint\vec{H}\cdot d\vec{l}=(H_{2t}-H_{1t})l=K_s l=I$$
$$=\int-\nabla U\cdot d\vec{l}=\int-dU=-\Delta U$$

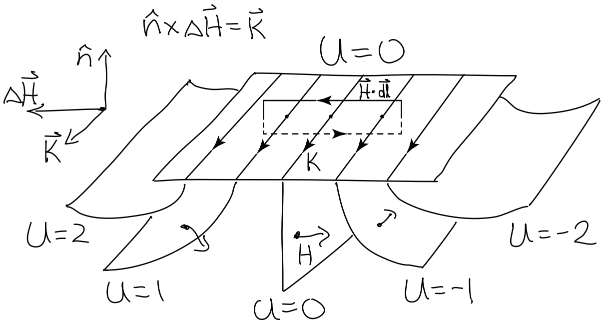
$$\boxed{-\Delta U=\mathcal{E}_H=I}$$

~ electric

	vector	component	potential
$\mathcal{E}_E=0$	$\hat{n}\times\Delta\vec{E}=0$	$E_{2t}=E_{1t}$	$\Delta V=0$
$\Phi_D=Q$	$\hat{n}\cdot\Delta\vec{D}=\sigma$	$D_{2n}-D_{1n}=\sigma$	$-\Delta\varepsilon\frac{\partial V}{\partial n}=\sigma$

~ magnetic

	vector	component	potential
$\mathcal{E}_H=I$	$\hat{n}\times\Delta\vec{H}=\vec{K}$	$H_{2t}-H_{1t}=K_s$	$-\Delta U=I$
$\Phi_B=0$	$\hat{n}\cdot\Delta\vec{B}=0$	$B_{2n}=B_{1n}$	$\Delta\mu\frac{\partial U}{\partial n}=0$



~ surface current flows along U equipotentials ~ U is a SOURCE potential

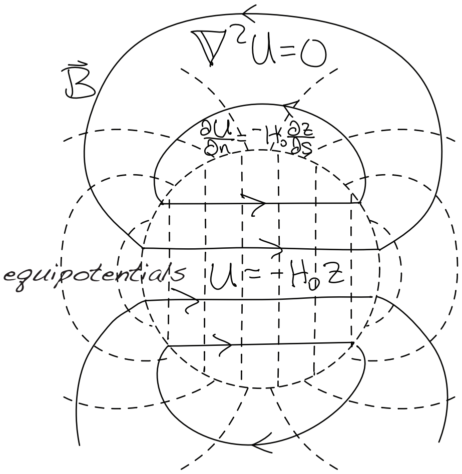
~ the current I=I2-I1 flows between any two equipotential lines U=I1 and U=I2

Scalar Potential Method

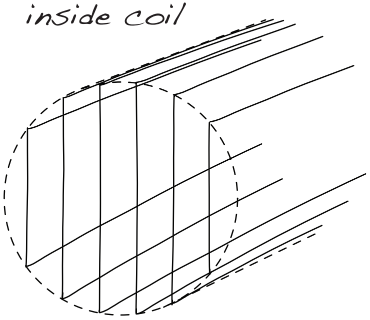
- * procedure for designing a coil based on required fields and geometry:
  - ~ solve $\nabla^2 U = 0$ with flux boundary conditions from known external fields
  - ~ draw the equipotential CURVES on the boundary to form the windings (wires)
  - ~ current through each wire = difference between adjacent equipotentials

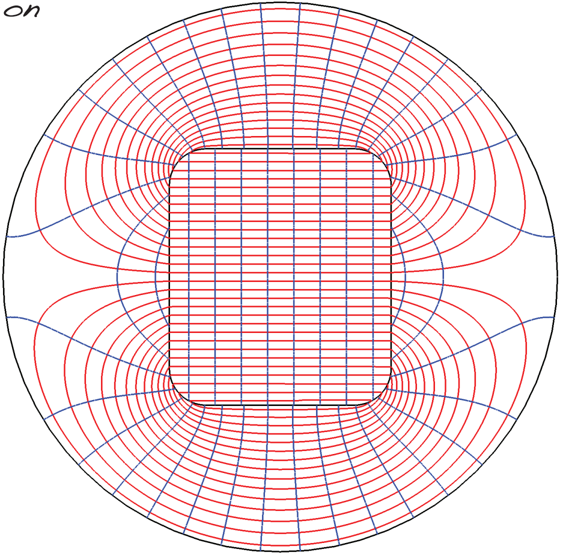
- * utility of electric and magnetic potentials - direct relation to physical devices
  - ~ it is only possible to control electric potential, NOT charge distribution in a conductor
  - ~ conversely, it IS easy to control current distributions (by placement of wires) but this is related to the magnetic scalar potential

- * example: cos-theta coil
  - ~ analog of cylindrical (2-d) electric dipole
  - ~ longitudinal windings, perfectly uniform field inside
  - ~ solve Laplace equation with flux boundary condition
  - ~ double cos-theta coil B=0 outside outer coil



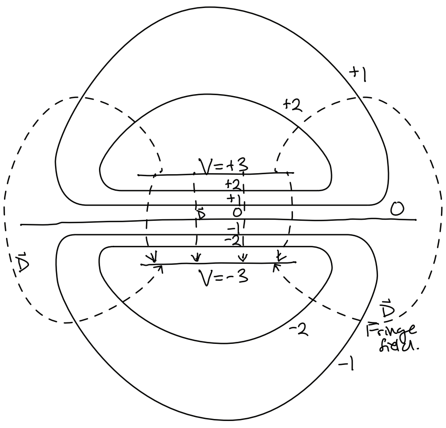
windings of inside coil



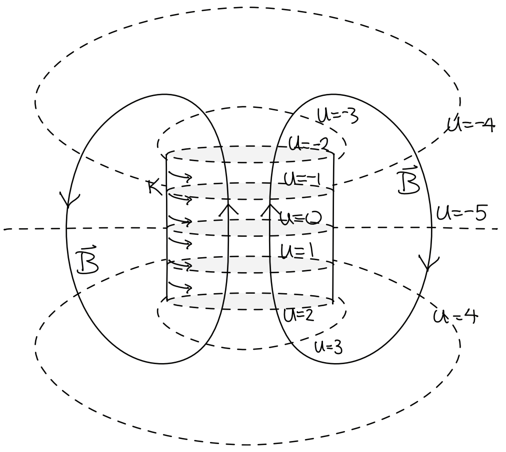


- * comparison of electrical and magnetic components

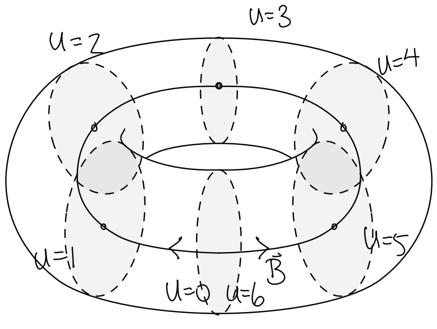
CAPACITOR



SOLENOID



TOROID

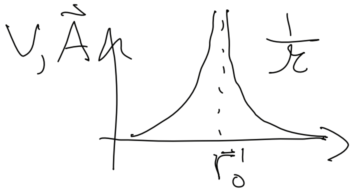


Section 5.4 - Magnetic Vector Potential

* Helmholtz theorem

$$\vec{B} = -\nabla\left(-\nabla^{-2}\cancelto{0}{\nabla\cdot\vec{B}}\right) + \nabla\times\overbrace{\left(-\nabla^{-2}\underbrace{\nabla\times\vec{B}}_{\mu_0\vec{J}}\right)}^{\vec{A}} \qquad \nabla\cdot\vec{B} = 0 \quad \rightleftharpoons \quad \vec{B} = \nabla\times\vec{A}$$

$$\vec{A} = -\nabla^{-2}\mu_0\vec{J} = \frac{\mu_0}{4\pi}\int\frac{\vec{J}'d\tau'}{r}$$



$$\nabla\times\vec{B} = \nabla\times(\nabla\times\vec{A}) = \nabla\,\cancelto{0}{\nabla\cdot\vec{A}} - \nabla^2\vec{A} = \mu_0\vec{J}$$

* Gauge invariance: A is NOT unique! Only $\nabla\times\vec{A}$ specified, not $\nabla\cdot\vec{A}$ (Helmholtz)

$\nabla\times\nabla\lambda = 0$ so $\vec{A} = \vec{A}_0 + \nabla\lambda$ also satisfies $\vec{B} = \nabla\times\vec{A}$

~ λ is called a "gauge transformation", the set of all λ's forms a mathematical group symmetry under gauge transformations is the basis of quantum field theories

~ a particular choice of A or a constraint on A is called a "gauge"

~ "Coulomb" or "radiation" gauge: $\nabla\cdot\vec{A} = 0$ always possible, unique up to B.C.'s if $\vec{B} = \nabla\times\vec{A}_0$ let $\vec{A} = \vec{A}_0 + \nabla\lambda$ and solve for $\lambda \ni \nabla\cdot\vec{A} = 0$. (another Poission eq.)

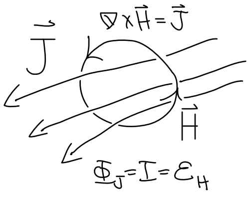
$$\nabla^2\lambda = -\nabla\cdot\vec{A}_0 \qquad \lambda = \frac{1}{4\pi}\int\frac{\nabla'\cdot\vec{A}_0'\,d\tau'}{r} = -\nabla^{-2}\nabla\cdot\vec{A}_0$$

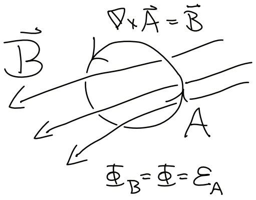
* Boundary conditions

H links current, A links flux

$$\mathcal{E}_H \equiv \oint_{\partial S}\vec{H}\cdot d\vec{l} = \int_S\nabla\times\vec{H}\cdot d\vec{a} = \int\vec{J}\cdot d\vec{a} = \Phi_J = I$$

$$\mathcal{E}_A \equiv \oint_{\partial S}\vec{A}\cdot d\vec{l} = \int_S\nabla\times\vec{A}\cdot d\vec{a} = \int_S\vec{B}\cdot d\vec{a} = \Phi_B$$





$$\left.\begin{array}{lll} \nabla\cdot\vec{A} = 0 & \Rightarrow \Phi_A = 0 & \hat{n}\cdot\Delta\vec{A} = 0 \\ \nabla\times\vec{A} = \vec{B} & \Rightarrow \mathcal{E}_A = \Phi_B & \hat{n}\times\Delta\vec{A} = \vec{0} \end{array}\right\} \quad \Delta\frac{\partial\vec{A}}{\partial t} = \Delta\frac{\partial\vec{A}}{\partial s} = 0 \qquad \Delta\vec{A} = \vec{0}$$

$$\left.\begin{array}{l} \nabla\cdot\vec{A} = 0 \Rightarrow \Delta\frac{\partial A_n}{\partial n} + \cancel{\Delta\frac{\partial A_t}{\partial t}} + \cancel{\frac{\partial A_s}{\partial s}} = 0 \qquad \Delta\frac{\partial A_n}{\partial n} = 0 \\ \hat{n}\times\Delta\vec{B} = \mu_0\vec{K} \Rightarrow \hat{n}\times(\hat{n}\,\partial_n + \cancel{\hat{s}\,\partial_s} + \cancel{\hat{t}\,\partial_t})\times\Delta\vec{A} = -\Delta\frac{\partial\vec{A}_\perp}{\partial n} = \mu_0\vec{K}_\perp \end{array}\right\} \quad \Delta\frac{\partial\vec{A}}{\partial n} = -\mu_0\vec{K}$$

* Summary of vector potential

$$\begin{tikzcd} \text{gauge} & \text{potential} & \text{field} & \text{source} & \\ \lambda \arrow[r, "d"] & (V,\vec{A}) \arrow[r, "d"] & (\vec{E},\vec{B}) \arrow[r, "d"] \arrow[d, phantom, "\varepsilon \| \mu \quad \text{Maxwell eq.'s}"] & 0 & \\ \text{gauge invariance} & & (\vec{D},\vec{H}) \arrow[r, "d"] & (\rho,\vec{J}) \arrow[r, "d"] & 0 \end{tikzcd}$$

Poisson's eq.   conservation of charge

$$\begin{tikzcd} \vec{A} \arrow[r, "\nabla\times"] \arrow[rr, bend right, "-\nabla^2"'] & \vec{B} \arrow[r, "\nabla\times"] & \mu_0\vec{J} \end{tikzcd}$$

Physical Significance of Vector Potential

- Physical significance: qV = potential energy qA = "potential momentum"
  - ~ it is the energy/momentum of interaction of a particle in the field
  - ~ some special cases can be solved using conservation of momentum, but you must account for momentum of the field unless there are no gradients
  - ~ $(V,\vec{A})$ is a 4-vector, like $(E,\vec{p})$ $(c,\vec{v})$ $(\rho,\vec{J})$
  - ~ $q(V-\vec{v}\cdot\vec{A})$ is a velocity-dependent potential

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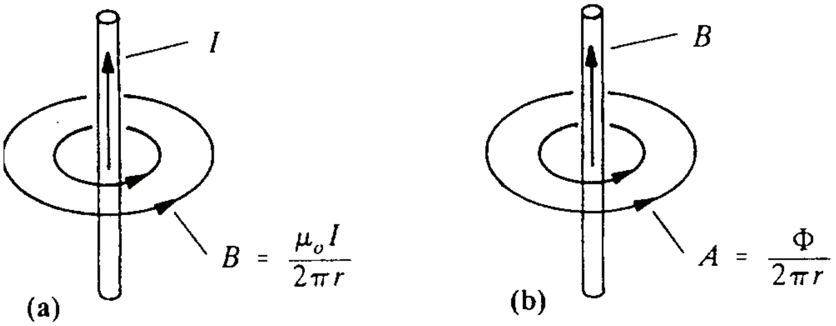
> The vector **A** represents in direction and magnitude the time integral [that is, impulse] of the electromagnetic intensity which a particle placed at the point (x,y,z) would experience if the primary current were suddenly stopped.

[2] J. C. Maxwell, *A Treatise on Electricity and Magnetism* (Oxford University, Oxford, 1873), 1st ed. Article 590.

- B flux tube (solenoid)

Wire	Solenoid	
$B=\frac{\mu_0 I}{2\pi s}$	$A=\frac{\Phi_B}{2\pi s}$	(inside)
$\vec{B}=\frac{\mu_0}{2}\vec{J}\times\vec{r}$	$\vec{A}=\frac{1}{2}\vec{B}\times\vec{r}$	(outside)

$\nabla\times\vec{H}=\vec{J}$ $\nabla\times\vec{A}=\vec{B}$



- Coaxial cable, straight conductor

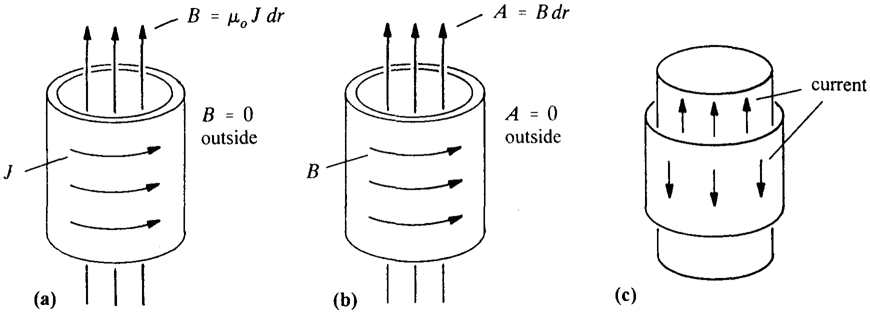
$$B=\mu_0 K=\mu_0\frac{I}{2\pi S}$$

$$A=B\,dr=\frac{\mu_0 I\,ds}{2\pi S}$$

$$A(r)=\frac{\mu_0 I}{2\pi}\left[\ln(b)-\ln(s)\right]$$

$$\rightarrow \frac{-\mu_0 I}{2\pi}\ln(s)$$

$$V(r)=\frac{\lambda}{2\pi\epsilon_0}\ln(s)$$



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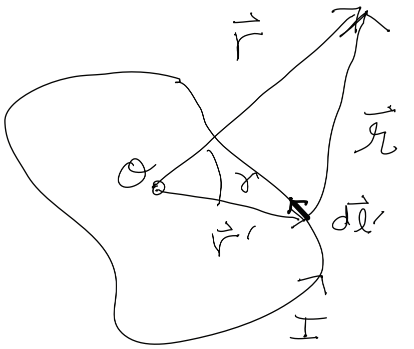
Section 5.4.3 - Multipole Expansion



* Similar to electrostatics, expand 1/r

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\oint\frac{d\vec{l}'}{\mathscr{r}} = \frac{\mu_0 I}{4\pi}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\oint r'^{l}P_l(\cos\gamma)\,d\vec{l}'$$

$$\frac{1}{\mathscr{r}} = \frac{1}{\sqrt{r^2-2rr'\cos\gamma+r'^2}} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\gamma)$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\Big[\frac{1}{r}\oint d\vec{l}' + \frac{1}{r^2}\oint r'\cos\gamma\, d\vec{l}$$
$$+\frac{1}{r^3}\oint r'^2\left(\frac{3}{2}\cos^2\gamma-\frac{1}{2}\right)d\vec{l}' + \dots\Big]$$

$$= \frac{\mu_0 I}{4\pi}\Big[\frac{1}{r}\cancel{\overset{\oint \text{loop}}{\oint d\vec{l}'}} + \frac{\vec{r}}{r^3}\cdot\oint d\vec{l}'\,\vec{r}\,' + \frac{\vec{r}}{r^5}\cdot\oint d\vec{l}'\left(\frac{3}{2}\vec{r}\,'\vec{r}\,'-\frac{1}{2}r'^2\right)\cdot\vec{r}$$

no monopole　　dipole　　quadrupole

$$\oint_{\partial S}\vec{v}\cdot d\vec{l} = \int_S\nabla\times\vec{v}\cdot d\vec{a} \qquad \text{(Stokes)}$$

divergence theorems

$$\boxed{\begin{aligned}\oint_{\partial S} d\vec{l}\,\dots &= \int_S d\vec{a}\times\nabla\,\dots\\ \int_{\partial R} d\vec{a}\,\dots &= \int_R d\tau\,\nabla\,\dots\end{aligned}}$$

let $\vec{v} = \vec{c}\,T$ then $\nabla\times\vec{v} = \nabla\times\vec{c}\,T = \nabla T\times\vec{c}$

$$\vec{c}\cdot\oint_{\partial S} T\,d\vec{l} = \int_S\nabla T\times\vec{c}\cdot d\vec{a} = \vec{c}\cdot\int_S d\vec{a}\times\nabla T$$

$$\oint_{\partial S} T\,d\vec{l} = -\int_S\nabla T\times d\vec{a}$$

let $T = \vec{c}\cdot\vec{r}$ then $\displaystyle\oint_{\partial S}\vec{c}\cdot\vec{r}\,d\vec{l} = -\int_S\underbrace{\nabla(\vec{c}\cdot\vec{r})}_{\vec{c}\times(\cancel{\nabla\times\vec{r}}) + \underbrace{(\vec{c}\cdot\nabla)\vec{r}}_{\vec{c}}}\times d\vec{a}$

$$\oint_{\partial S}\vec{c}\cdot\vec{r}\,d\vec{l} = -\int\vec{c}\times d\vec{a}$$

$$\oint\hat{r}\cdot\vec{r}\,'\,d\vec{l}' = -\hat{r}\times\int d\vec{a}'$$

$\displaystyle\int_V\nabla T\,d\tau = \oint_{\partial V}d\vec{a}\,T$　　$\displaystyle\oint_{\partial V}d\vec{a} = 0$ if $T=1$　so $\displaystyle\int_{S_1}d\vec{a} = \int_{S_2}d\vec{a}$ if $\partial S_1 = \partial S_2$

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3} \qquad \vec{m}\equiv\int I\,d\vec{a} = I\vec{a}$$

compare: $\times\leftrightarrow\cdot$　$V_{dip}(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\vec{p}\cdot\vec{r}}{r^3}$

* in spherical coordinates,

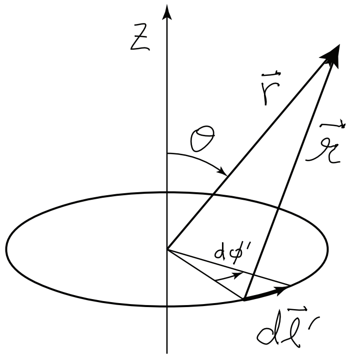
$$\vec{B} = \nabla\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi}\\ \partial_r & \partial_\theta & \partial_\phi\\ A_r & rA_\theta & r\sin\theta A_\phi\end{vmatrix} \qquad \vec{A} = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi} = A_\phi\hat{\phi}$$

$$= \frac{\mu_0 m}{4\pi}\frac{1}{r^2\sin\theta}\left[\hat{r}\,\partial_\theta\left(r\frac{\sin^2\theta}{r^2}\right) - r\hat{\theta}\,\partial_r\left(r\sin^2\theta A_\phi\right)\right]$$

$$\vec{B}_{dip}(\vec{r}) = \frac{\mu_0 m}{4\pi r^3}\left[2\cos\theta\,\hat{r}+\sin\theta\,\hat{\theta}\right]$$

compare: $\dfrac{p}{4\pi\varepsilon_0}\leftrightarrow\dfrac{\mu_0 m}{4\pi}$　$\vec{E} = \dfrac{p}{4\pi\varepsilon_0 r^3}\left[2\cos\theta\,\hat{r}+\sin\theta\,\hat{\theta}\right]$

* Example: current loop dipole



$r\hat{\theta} = -x\hat{z} + z\hat{x}$ since $y=0$

$r'^2 = x'^2 + y'^2$

$$\vec{\mathscr{r}} = (\vec{r} - \vec{r}\,') = (x\hat{x} + z\hat{z}) - (x'\hat{x} + y'\hat{y})$$

$$d\vec{l}' = d\vec{r}\,' = d(r'\cos\phi'\,\hat{x} + r'\sin\phi'\,\hat{y}) = r'd\phi'\hat{\phi}'$$

$$= -r'\sin\phi'\,\hat{x} + r'\cos\phi'\,\hat{y}\;\; d\phi' = (-y'\hat{x} + x'\hat{y})d\phi'$$

$$d\vec{l}' \times \vec{\mathscr{r}} = (-y'\hat{x} + x'\hat{y}) \times (x\hat{x} + z\hat{z} - x'\hat{x} - y'\hat{y})\; d\phi'$$

$$= zy'\hat{y} + y'^2\hat{z} \;\; - x'x\hat{z} \;\; + x'z\hat{x} \;\; + x'^2\hat{z}\;\; d\phi'$$

$$= (zy'\hat{y} + r'^2\hat{z} + rx'\hat{\theta})d\phi'$$

$$\vec{r}\cdot\vec{r}\,' = (x\hat{x} + z\hat{z})\cdot(x'\hat{x} + y'\hat{y}) = xx'$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\oint \frac{I'd\vec{l}' \times \vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{\mu_0 I'}{4\pi}\int_{\phi=0}^{2\pi} \frac{\cancel{zy'}\hat{y} + r'^2\hat{z} + rx'\hat{\theta}}{(r^2 + xx' + r'^2)^{3/2}}\, d\phi'$$

~ the above integral is antisymmetric under $y \leftrightarrow y'$ first term vanishes

~ to get the dipole approximation, assume $r' \ll r$

$$\mathscr{r}^{-3} = \left(r^2\left(1 + 2\frac{xx'}{r^2} + \dots\right)\right)^{-3/2} = r^{-3}\left(1 + 3\frac{xx'}{r^2} + \dots\right)$$ binomial expansion

$$\vec{B}(\vec{r}) \approx \frac{\mu_0 I'}{4\pi r^3}\int_{\phi=0}^{2\pi} (r'^2\hat{z} + rx'\hat{\theta})\left(1 + 3\frac{xx'}{r^2}\right)d\phi'$$ order by powers of r' or x'

$$= \frac{\mu_0 I}{4\pi r^3}\int_{\phi'=0}^{2\pi} rx'\hat{\theta} + \left(r'^2\hat{z} + 3\frac{xx'^2}{r}\hat{\theta}\right) + \mathcal{O}(r'^3)$$

~ the first term $= \int \cos\phi' d\phi' = 0$ - no monopole! $\int \cos^2\phi d\phi \sim \int \frac{1}{2} d\phi$

~ the second two terms are the dipole $\vec{m}' = I\vec{a} = (\pi r'^2)I'\hat{z}$

$$\vec{B}(\vec{r}) = \frac{\mu_0 I r'^2}{2r^4}\left(r\hat{z} + \frac{3}{2}x\hat{\theta}\right) = \frac{\mu_0}{4\pi}\,\frac{3\vec{m}\cdot\hat{r}\,\hat{r} - \vec{m}}{r^3}$$

$r\hat{z} = z\hat{r} - x\hat{\theta}$ since $y=0$

~ equivalent to electric dipole under correspondence $\frac{\vec{p}}{4\pi\varepsilon_0} \longleftrightarrow \frac{\mu_0\vec{m}}{4\pi}$

Section 6.1 - Magnetization

* review: development of electric and magnetic multipole potentials

$$\nabla\times\vec{E}=0 \Rightarrow \vec{E}=-\nabla V \qquad \nabla\cdot\vec{B}=0 \Rightarrow \vec{B}=\nabla\times\vec{A}$$

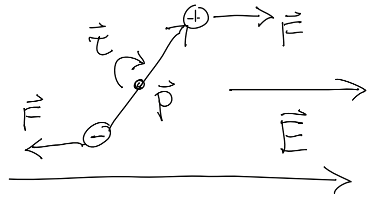
$$\nabla\cdot\vec{E}=\rho/\varepsilon_0 \Rightarrow \nabla^2 V=-\rho/\varepsilon_0 \qquad \nabla\times\vec{B}=\mu_0\vec{J} \Rightarrow \nabla\times(\nabla\times\vec{A})=\mu_0\vec{J}$$

$$=\nabla(\cancelto{0}{\nabla\cdot\vec{A}})-\nabla^2\vec{A}$$

$$V=\frac{1}{4\pi\varepsilon_0}\int\frac{dq}{\mathscr{r}}=\frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r}+\frac{\vec{p}\cdot\vec{r}}{r^3}+\cdots\cdots\right) \qquad \vec{A}=\frac{\mu_0}{4\pi}\int\frac{d\vec{q}}{\mathscr{r}}=\frac{\mu_0}{4\pi}\left(\frac{1}{r}\cancelto{0}{\int d\vec{q}}+\frac{\vec{m}\times\vec{r}}{r^3}\cdots\cdots\right)$$

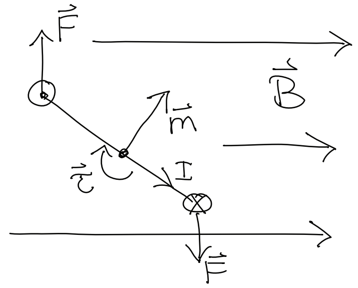
$$\vec{p}=\int dq\,\vec{r} \qquad dq=\rho\,d\tau \qquad \vec{m}=\frac{1}{2}\int\vec{r}\times d\vec{q} \qquad d\vec{q}=\vec{J}\,d\tau$$

* dynamics of dipoles in fields (compare Electric and Magnetic)



$$\vec{\tau}_e=\vec{p}\times\vec{E}$$

$$W=-\int\tau\,d\theta$$
$$=-\vec{p}\cdot\vec{E}$$



$$\vec{\tau}_m=\vec{m}\times\vec{B}$$

$$W=-\int\tau\,d\theta$$
$$=-\vec{m}\cdot\vec{B}$$

$$\vec{F}=\int dq\,\vec{E}=\int dq\,(1+\vec{r}\cdot\nabla)\vec{E} \qquad \vec{F}=\int d\vec{q}\times\vec{B}=\int d\vec{q}\,(\cancelto{0}{1}+\vec{r}\cdot\nabla)\vec{B}$$

$$=Q+(\vec{p}\cdot\nabla)\vec{E} \qquad =(\vec{m}\times\nabla)\times\vec{B}$$

$$\vec{p}=\int dq\,\vec{r} \qquad dq=\rho\,d\tau \qquad \vec{m}=\frac{1}{2}\int\vec{r}\times d\vec{q} \qquad d\vec{q}=\vec{J}\,d\tau$$

$$\nabla(\vec{p}\cdot\vec{E})=(\vec{p}\cdot\nabla)\vec{E}+\vec{p}\times(\cancelto{0}{\nabla\times\vec{E}}) \qquad \nabla(\vec{m}\cdot\vec{B})=(\vec{m}\times\nabla)\times\vec{B}-\vec{m}(\cancelto{0}{\nabla\cdot\vec{B}})$$

$$\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-C(\vec{A}\cdot\vec{B}) \qquad (\vec{A}\times\vec{B})\times\vec{C}=\vec{B}(\vec{A}\cdot\vec{C})-\vec{A}(\vec{B}\cdot\vec{C})$$

$$\vec{a}\times(\nabla\times\vec{b})=\overset{\curvearrowright}{\nabla\,\vec{a}\cdot\vec{b}}-\vec{b}\,\underbrace{\vec{a}\cdot\nabla} \qquad (\vec{a}\times\nabla)\times\vec{b}=\overset{\curvearrowright}{\nabla(\vec{a}\cdot\vec{b})}-\vec{a}(\nabla\cdot\vec{b})$$

$$\overset{\curvearrowright}{\nabla\vec{b}}\cdot\vec{a}=(\vec{a}\cdot\nabla)\vec{b}+\vec{a}\times(\nabla\times\vec{b}) \qquad = \qquad (\vec{a}\times\nabla)\times\vec{b}+\vec{a}(\nabla\cdot\vec{b})$$

Section 6.2 - Field of Magnetized Object

* polarizability

electric $\vec{p} = \alpha \vec{E}$	magnetic $\vec{m} = \beta \vec{H}$	
a) stretch +/- charge	a) torque on spin	paramagnetic
b) torque on permanent dipoles	b) speed up orbitalsv	diamagnetic
<- compare/contrast ->	c) self-alignment of dipoles	ferromagnetic

* diamagnetism in the atom

~ magnetic dipole increases / decreases in response to changing magnetic field

~ not completely induced dipole moment like electric case

$$m = Ia = \frac{e\omega}{2\pi} \cdot \pi r^2 = \frac{e}{2m_e} \cdot m_e r^2 \omega = \frac{e}{2m_e} L$$

$$\vec{m} = \gamma \vec{L} \qquad \gamma = \frac{e}{2m_e} \qquad \mu_B = \gamma \hbar$$

gyromagnetic ratio Bohr magneton

~ by Lens' law, L and therefore m adjust to counteract the change in field

* magnetization $\vec{M} = \frac{1}{\tau} \int_V d\tau \, \vec{m}$ $\qquad d\vec{m} = \vec{M} d\tau$

* field of a magnetized object: bound currents

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0}{4\pi} \int' \frac{M' \times \hat{\mathscr{r}} \, d\tau'}{\mathscr{r}^2} = \frac{\mu_0}{4\pi} \int' M' \times \nabla' \frac{1}{\mathscr{r}} \, d\tau'$$

~ generalized divergence theorem $\int d\tau \, \nabla \circ \equiv \oint d\vec{a} \, \circ$ $\qquad \circ = \cdot \, , \times \, ,$ or scalar mult.

$$\oint' d\vec{a}' \times \frac{M'}{\mathscr{r}} = \int' d\tau' \nabla' \times \frac{M'}{\mathscr{r}} = \int' d\tau' \, \frac{\nabla' \times M'}{\mathscr{r}} - M' \times \nabla' \frac{1}{\mathscr{r}}$$

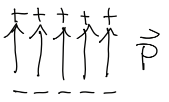
$$\vec{A} = \frac{\mu_0}{4\pi} \int' \frac{\vec{J}_b'}{\mathscr{r}} d\tau + \frac{\mu_0}{4\pi} \oint \frac{\vec{K}_b}{\mathscr{r}} da$$

where $\vec{J}_b = \nabla \times \vec{M}$ $\qquad \vec{K}_b = - \hat{n} \times \vec{M}$

compare $\rho_b = - \nabla \cdot \vec{P}$ $\qquad \sigma_b = \hat{n} \cdot \vec{P}$

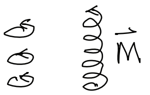
~ notice the difference in signs

* physical model of polarization vs. magnetization

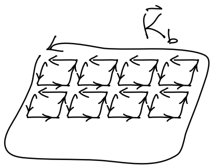


P is extensive transversely

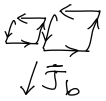
$\vec{p}$ -chain



M is extensive longitudinally



$\vec{M}$ ~ flow

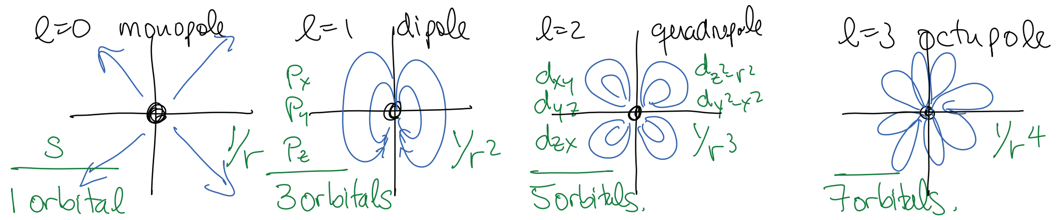


Section 6.3 – Auxiliary Field $\mathcal{H}$

* reminder – Multipoles: gerneral solution to $\nabla^2 V = 0$ with azimuthal symmetry

$$V = \frac{1}{4\pi\varepsilon_0} \sum_\ell \Big(\underbrace{Q_\ell^{ext} r^\ell}_{V_{int}} + \underbrace{Q_\ell^{int} r^{-\ell-1}}_{V_{ext.}} \Big) P_\ell(c_\theta) \qquad \frac{1}{4\pi\varepsilon_0} \rightarrow \frac{\mu_0}{4\pi} \quad V \rightarrow U \qquad Q_\ell \rightarrow \begin{matrix} E_\ell \\ M_\ell \end{matrix} \text{ multipole moments.}$$

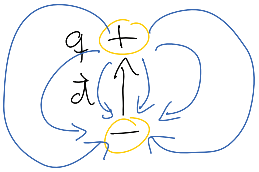
* magnetic multipoles



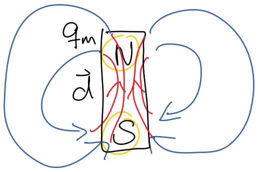
default $\longrightarrow$ if $q=0$ $\longrightarrow$ if $\vec{p}, \vec{m} = 0$ $\longrightarrow$ if $\overleftrightarrow{Q} = 0$

$\boxed{q}$ $\quad \vec{p} = q\vec{d} \quad \vec{m} = \boxed{I}\vec{a} \quad \overleftrightarrow{Q} = \vec{p}\vec{d} + \vec{d}\vec{p} - 2\vec{p}\cdot\vec{d}\,\overleftrightarrow{I} \quad \ldots$

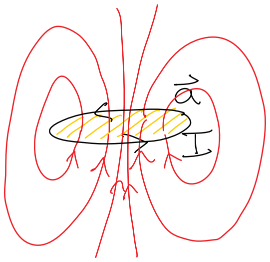
* compare / contrast $\vec{p} = q\vec{d}$ vs. $\vec{m} = I\vec{a}$



Du Fay



Gilbert



Ampère

* polarization $\vec{P}$, magnetization $\vec{M}$ = dipole density

$$\boxed{\begin{matrix} V = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}\cdot\vec{r}}{r^3} & \vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m}\times\vec{r}}{r^3} \\ \vec{p} \rightarrow d\vec{p} = \vec{P} d\tau & \vec{m} \rightarrow d\vec{m} = \vec{M} d\tau \\ \rho_b = -\nabla\cdot\vec{P} & \vec{J}_b = \nabla\times\vec{M} \end{matrix}}$$

~ to get effective charge / current distribution:

a) expand V into dipole potential

b) integrate V due to dipole density field

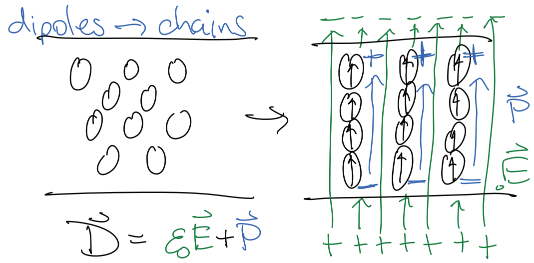
c) $\hat{r}/r^2 = -\nabla 1/r$, integrate by parts

d) compare with $V = \int \frac{\rho\, d\tau}{4\pi r} + \int \frac{\sigma\, da}{4\pi r}$ to get ρ_b, $\vec{J}_b$

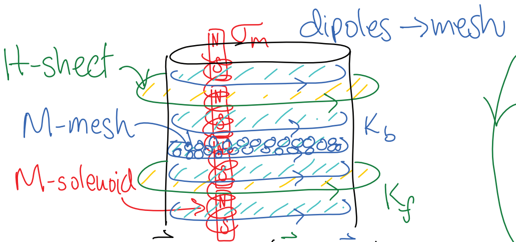
~ can also expand magnetic scalar potential to get "magnetic pole density" directly analogous with electric charge $\quad \rho_m = -\nabla\cdot\vec{M}$

~ given permanent $\vec{P}$ or $\vec{M}$, use ρ_b, ρ_m or $\vec{J}_m$ to calculate fields

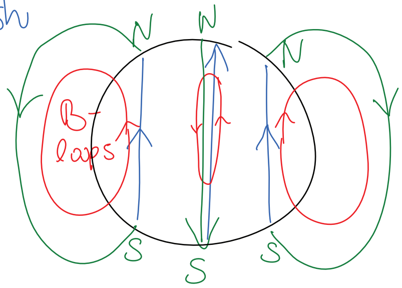
* polarization chain; magnetization solenoid and mesh



$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$



$$\vec{B} = \mu_0(\vec{H} + \vec{M})$$



* source fields D, H only include free charge/current as sources

$$\nabla \cdot \varepsilon_0 \vec{E} = \rho_f + \rho_b$$
$$\nabla \cdot \vec{P} = -\rho_b$$
$$\boxed{\begin{aligned}\nabla \cdot \vec{D} &= \rho_f \\ \nabla \times \vec{E} &= \vec{0}\end{aligned}}$$

$$\nabla \times \vec{B}/\mu_0 = \vec{J}_f + \vec{J}_b$$
$$- \quad \nabla \times M = \vec{J}_b$$
$$\boxed{\begin{aligned}\nabla \times \vec{H} &= \vec{J}_f \\ \nabla \cdot \vec{B} &= 0\end{aligned}}$$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon \vec{E}$$
$$\varepsilon = \varepsilon_0 \varepsilon_r = \varepsilon_0(1+\chi_e)$$

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu \vec{H}$$
$$\mu = \mu_0 \mu_r = \mu_0(1+\chi_m)$$

* boundary conditions: $\nabla \to \hat{n}\Delta \quad d\tau \to da$

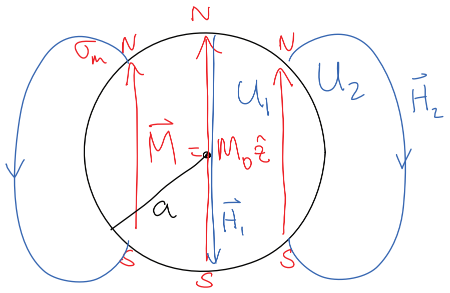
~ don't double count bound charge: K_b, σ_m, μ_r all account for the same thing!

fields	potentials	fields	potentials
$\hat{n} \times \Delta\vec{E} = 0$	$\Delta V = 0$	$\hat{n} \times \Delta\vec{H}_t = \vec{K}_s$	$\Delta \mathcal{U} = -I \quad -\Delta \frac{\partial \mathcal{U}}{\partial t} = K_s$
$\hat{n} \cdot \Delta\vec{D} = \sigma$	$-\Delta \varepsilon \frac{\partial V}{\partial n} = \sigma$	$\hat{n} \cdot \Delta\vec{B} = \mu_0 \sigma_m$	$-\Delta \mu_r \frac{\partial \mathcal{U}}{\partial n} = \sigma_m \quad \hat{n} \times \hat{t} = \hat{s}$

* three ways to solve similar magnetic boundary value problems:

a) use Gilbert "pole density" ρ_m, σ_m explicitly ~ see Example #1
b) use Ampere "bound current" $\vec{J}_b$ explicitly ~ see Example #2
c) absorb magnetization into "permeability" μ ~ see Example #3

* Example 1: Magnetic pole density σ_m



$$\nabla\times\vec{H} = \vec{J}_f = \vec{0} \quad \rightarrow \quad H = -\nabla U$$

$$\nabla\cdot\vec{B} = \nabla\cdot\mu_0(\vec{H}+\vec{M}) = 0 \quad \rightarrow \quad -\nabla^2 U = \rho_m \quad \text{no } \mu_0!$$

where $\rho_m = -\nabla\cdot\vec{M}$ $\sigma_m = \hat{n}\cdot\vec{M} = M_0\cos\theta$

B.C.'s: $U_1 = U_2$ $-\Delta\frac{\partial U}{\partial n} = \sigma_m$

$$\vec{m} = \int\vec{M}\,d\tau = \tfrac{4}{3}\pi a^3\vec{M}$$

$$U_1 = \sum_{\ell=0}^{\infty}\left[A_\ell\,(r/a)^\ell + \cancelto{0}{B_\ell}(a/r)^{\ell+1}\right]P_\ell(c_\theta)$$

$$U_2 = \sum_{\ell=0}^{\infty}\left[\cancelto{0}{C_\ell\,(r/a)^\ell} + D_\ell\,(a/r)^{\ell+1}\right]P_\ell(c_\theta)$$

BC#1: $U_1\big|_{r=a} = U_2\big|_{r=a} \quad \rightarrow \quad A_\ell = D_\ell$

BC#2: $$-\frac{\partial U_2}{\partial r}\Big|_a + \frac{\partial U_1}{\partial r}\Big|_a = \sum_\ell A_\ell\Big[\underbrace{-(\ell+1)(\tfrac{-1}{a}) + (\ell)(a)}_{2\ell+1/a}\Big]\overbrace{P_\ell(c_\theta)}^{\hat{e}_\ell} = M_0\overbrace{P_1(c_\theta)}^{\hat{e}_1}$$

$$A_0 = A_2 = A_3 = \dots = 0 \qquad A_1 = \tfrac{a}{3}M_0$$

$$U_1 = \tfrac{1}{3}M_0\,r\cos\theta = \tfrac{1}{3}M_0 z$$
$$\vec{H}_1 = -\vec{M}/3$$
$$\vec{B}_1 = \mu_0(\vec{H}+\vec{M}) = \tfrac{2}{3}\mu_0\vec{M}$$

$$U_2 = \frac{a^3}{3}M_0\,\frac{\cos\theta}{r^2} = \frac{1}{4\pi}\,\frac{\vec{m}\cdot\vec{r}}{r^3} \quad \text{dipole}$$
$$\vec{H}_2 = \frac{1}{4\pi r^3}(3\hat{r}\hat{r}\cdot\vec{m}-\vec{m}) \equiv \frac{\mathbb{Q}\vec{m}}{4\pi r^3}$$

see probs: 3.33 p155
5.33 p246

$$\vec{B}_2 = \mu_0\vec{H}_2$$

where $\mathbb{Q} \equiv 3\hat{r}\hat{r}\cdot - \mathbb{I}$

$\lim_{a\to 0}$: $$\boxed{\vec{E} = \frac{\mathbb{Q}\vec{p}}{4\pi\varepsilon_0 r^3} - \frac{1}{3\varepsilon_0}\vec{p}\,\delta^3(\vec{r})}$$

see prob 3.42 p157

$$\vec{H} = \frac{\mathbb{Q}\vec{m}}{4\pi r^3} - \tfrac{1}{3}\vec{m}\,\delta^3(\vec{r})$$

$$\vec{D} = \frac{\mu_0\mathbb{Q}\vec{p}}{4\pi r^3} + \frac{2}{3}\vec{p}\,\delta^3(\vec{r})$$

$$\boxed{\vec{B} = \frac{\mu_0\mathbb{Q}\vec{m}}{4\pi r^3} + \frac{2}{3}\mu_0\vec{m}\,\delta^3(\vec{r})}$$

see prob. 5.59 p 253

if $\vec{m} = \int d\tau\vec{M}$ is fixed as $a\to 0$ then
$\vec{M} \rightarrow \vec{m}\,\delta^3(\vec{r})$

note closed $\vec{B} = \mu_0(\vec{H}+\vec{M})$ lines of flux.

* *Example 2: Bound current density $\vec{J}_b$*

$\nabla \times \vec{H} = \vec{J}_{tot} = 0$ except on ∂ $\rightarrow$ $\vec{H} = -\nabla U$ discontinuity of U from

$\nabla \cdot \vec{B} = -\nabla \cdot \mu_0 \nabla U = 0$ $\rightarrow$ $\nabla^2 U = 0$ K_b on the boundary

$\vec{J}_b = \nabla \times \vec{M} = 0$ $\vec{K}_b = -\hat{n} \times \vec{M} = M_0 \sin\theta\, \hat{\phi}$ (treat K_b as a free current)

BC's: $\Delta \frac{\partial U}{\partial n} = 0$ $-\Delta \frac{\partial U}{\partial t} = K_s$

$$U_1 = \sum_{\ell=0}^{\infty} \left[A_\ell (r/a)^\ell + \cancel{B_\ell (a/r)^{\ell+1}} \right] P_\ell(c_\theta)$$

$$U_2 = \sum_{\ell=0}^{\infty} \left[\cancel{C_\ell (r/a)^\ell} + D_\ell (a/r)^{\ell+1} \right] P_\ell(c_\theta)$$

$$\text{BC\#1: } -\frac{\partial U_2}{\partial r}\Big|_a + \frac{\partial U_1}{\partial r}\Big|_a = \sum_{\ell=0}^{\infty} \left[-D_\ell (\ell+1)\left(\frac{-1}{a}\right) + A_\ell(\ell)\left(\frac{1}{a}\right) \right] P_\ell(c_\theta) = 0 \qquad D_\ell = \frac{-\ell}{\ell+1} A_\ell$$

$$\text{BC\#2: } -\frac{\partial U_2}{r\partial\theta}\Big|_a + \frac{\partial U_1}{r\partial\theta}\Big|_a = \sum_{\ell=0}^{\infty} \left[-D_\ell + A_\ell \right] \frac{1}{a} P_\ell'(c_\theta)(-\sin\theta) = M_0 \sin\theta$$

note: $P_\ell'(c_\theta)$ form a basis, and $P_1'(x) = 1$ like the RHS.

so $(-D_1 + A_1) \cdot \frac{-1}{a} = \left(\frac{-1}{2} - 1\right) A_1/a = M_0$ $\qquad A_1 = -\frac{2}{3} a M_0 = -2 D_1$

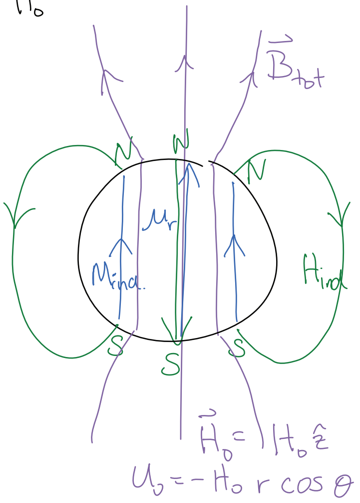
$U_1 = -\frac{2}{3} M_0 z$ $\qquad U_2 = \frac{1}{3} M_0 \frac{\cos\theta}{r^2} = \frac{\vec{m} \cdot \vec{r}}{4\pi r^3}$

$\vec{B}_1 = \mu_0 \vec{H}_1 = \mu_0 \frac{2}{3} \vec{M}$ $\qquad \vec{B}_2 = \frac{Q \vec{m}}{4\pi r^3}$

* note: $\vec{B} = \mu_0 \vec{H}$, not $\mu_0(\vec{H} + \vec{M})$ inside because we replaced $\vec{M}$ with its effective current distribution K_b

* thus $\vec{H}$ is different from Ex#1, but $\vec{B}$ is still the same.

* Example 3: Magnetic permeability (linear homogeneous isotropic material)
Permeable sphere in an external constant field H_0



$\vec{M} = \chi_m\vec{H}$ encapsulated in $\vec{B} = \mu\vec{H}$ see Ex4.7 p186

$\nabla\times\vec{H} = \vec{J}_{tot} = 0$ everywhere → $\vec{H} = -\nabla U$

$\nabla\cdot\vec{B} = -\nabla\cdot\mu\nabla U = 0$ → $\nabla^2 U = 0$

BC's: $U(\infty)\to -H_0 z$, $\Delta U = 0$, $\Delta\mu\frac{\partial U}{\partial n} = 0$

$$U_1 = \sum_{\ell=0}^{\infty}\left[A_\ell (r/a)^\ell + \overset{0}{\cancel{B_\ell}}(a/r)^{\ell+1}\right]P_\ell(c_\theta)$$

$$U_2 = \sum_{\ell=0}^{\infty}\left[C_\ell (r/a)^\ell + D_\ell (a/r)^{\ell+1}\right]P_\ell(c_\theta)$$

BC#0: $\lim_{r\to\infty} U_2(r) = \sum_{\ell=0}^{\infty} C_\ell (r/a)^\ell P_\ell(c_\theta) = -H_0 r c_\theta$ $\quad C_1 = -H_0 a$

we exclude all multipoles except $\ell=1$, since the source $\vec{H}_0$ is pure dipole.

BC#1: $U_2|_a - U_1|_a = (-H_0 a + D_1 - A_1) = 0 \quad\Rightarrow\quad D_1 = A_1 + H_0 a$

BC#2: $-\frac{\partial U_2}{\partial r}\Big|_a + \mu_r\frac{\partial U_1}{\partial r}\Big|_a = \left(H_0 + D_1\cdot 2/a + \mu_r A_1/a\right)\cos\theta = 0$

$3H_0 + (2+\mu_r)A_1/a = 0 \quad A_1/a = \frac{-3}{2+\mu_r}H_0 \quad D_1/a = \frac{-1+\mu_r}{2+\mu_r}H_0 = \frac{\chi_m}{2+\mu_r}H_0$

$U_1 = U_0 + \frac{\chi_m}{2+\mu_r}H_0 z = \frac{-3}{2+\mu_r}H_0 z \qquad \vec{H}_1 = \frac{3}{2+\mu_r}\vec{H}_0$

$U_2 = U_0 + \frac{\chi_m}{2+\mu_r}H_0 a^3\frac{\cos\theta}{r^2}$ where $U_0 = -H_0 z$

* compare with Ex#1: $U_1 = U_0 + \frac{1}{3}Mz \qquad U_2 = U_0 + \frac{1}{3}a^3 M\frac{\cos\theta}{r^2}$

$\vec{M} = \frac{3\chi_m}{2+\mu_r}\vec{H}_0 = \chi_m\left(\frac{3}{2+\mu_r}\right)\vec{H}_0 = \chi_m\vec{H}_1$ as dictated by $\vec{M} = \chi_m\vec{H}$

$\vec{B}_1 = \frac{3\mu}{2+\mu_r}\vec{H}_0 \qquad \vec{B}_2 = \mu_0\vec{H}_2$
$= \mu_0\left(\vec{H}_0 + \frac{\mathbb{Q}\vec{m}}{4\pi r^3}\right)$

where $\vec{m} = \frac{4}{3}\pi a^3\cdot\frac{3\chi_m}{2+\mu_r}H_0$
$\mathbb{Q}\vec{m} = 3\hat{r}\hat{r}\cdot\vec{m} - \vec{m}$

Section 6.4 - Magnetic Media

* constitutive relations: magnetic susceptibility and permeability

$$\varepsilon_0 \vec{E} = \vec{D} - \vec{P} \qquad \vec{D} = \varepsilon_0 (\vec{E} + \vec{P}) = \varepsilon_0 (1 + \chi_e) \vec{E} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E}$$

$$\frac{1}{\mu_0} \vec{B} = \vec{H} + \vec{M} \qquad \vec{B} = \mu_0 (\vec{H} + \vec{M}) = \mu_0 (1 + \chi_m) \vec{H} = \mu_0 \underbrace{\mu_r}_{K_m} \vec{H} = \mu \vec{H}$$

* linear and nonlinear media: $\vec{D}(\vec{E}, \vec{r}, \omega, T, \ldots)$ $\vec{M}(\vec{B}, \vec{r}, \omega, T, \ldots \text{history!})$

~ linear $\vec{M} = \overleftrightarrow{\mu} \cdot \vec{H}$ $\overleftrightarrow{\mu}(\vec{H}) = \text{const.}$ permeability independent of field strength

~ isotropic $\overleftrightarrow{\mu} = \mu \cdot \overleftrightarrow{I}$ $\vec{M} \parallel \vec{H}$ same permeability in all directions

~ homogeneous $\mu(\vec{r}) = \text{const.}$ same permeability throughout material
still dicontinuities at boundary

* Gaussian units (CGS) $\mu_0 = \varepsilon_0 = 1$ so $\mu = \mu_r$ $\varepsilon = \varepsilon_r$

~ $[H]$ = Oersted, $[B]$ = Gauss ~ 0.0001 Tesla

~ Units of E and B also the same! $\vec{F} = q(\vec{E} + \vec{v}/c \times \vec{B})$

* diamagnetism

~ most similar to electiric $\chi_m < 0$ $\sim -10^{-5}$

~ useful for levitation

~ superconductor (SC) $\chi_m = -1$! $\vec{B} = 0$

* paramagnetism $\chi_m > 0$ $\approx 10^{-5} - 10^{-1}$

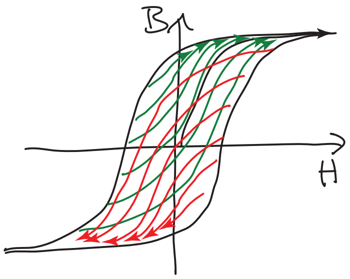
* ferromagnetism $\mu_r \gg 1$ $\Delta \vec{B} = \mu_\Delta \Delta \vec{H} (\vec{H}, \vec{B}, \vec{r}, \omega, T, \pm \text{direction})$

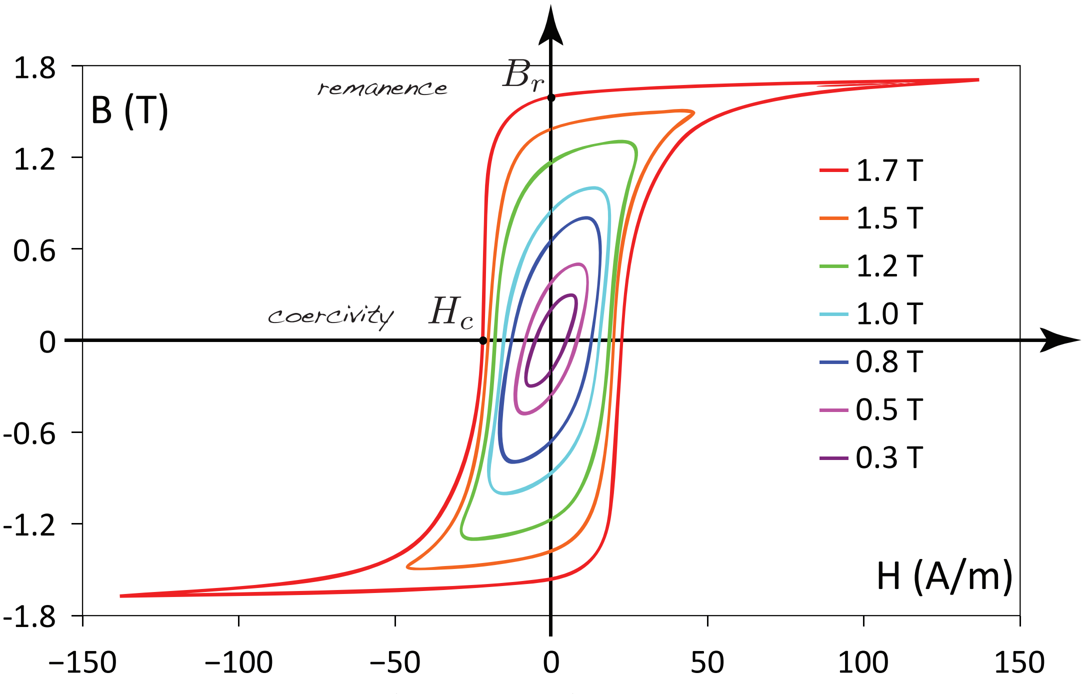
~ electromagnet

~ iron-core transformers

~ μ-metal

$\sim 10^3$ reduction in field.





http://en.wikipedia.org/wiki/Hysteresis

Section 7.1 - Electromotive Force

* review

~ current element

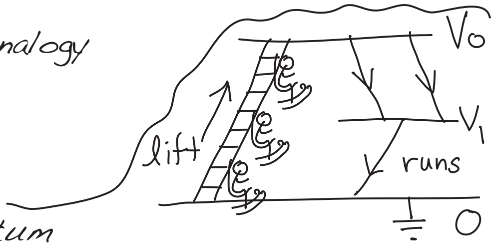
$$\begin{tikzcd} q \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \lambda \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \sigma \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \rho \arrow[d, leftrightarrow] \\ q\vec{v} \arrow[r, leftrightarrow] & I \arrow[r, leftrightarrow] & K \arrow[r, leftrightarrow] & J \end{tikzcd}$$

~ continuity $\frac{\partial}{\partial t}\rho + \nabla\cdot\vec{J} = 0$ conservation of charge

~ potential $\vec{E} = -\nabla V$ conservation of energy

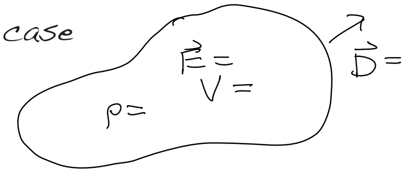
$\vec{B} = \nabla\times\vec{A}$ conservation of momentum

* ski lift analogy



* conductors

~ static case



~ if $\Delta V \neq 0$ then $m\vec{a} = \vec{F} = q\vec{E}$
leads to steady state current
$\vec{J} = \sigma\vec{E}$ (third constitutive equation)

~ this law depends on material properties
for example, a vacuum tube obeys
the nonlear Child-Lamgmuir law $I = KV^{3/2}$
Thermionic emmission depends on temp.

~ terminal (drift) velocity in a conductor

$$b\vec{v}_d = -\vec{F}_f = \vec{F}_e = q\vec{E}$$

$$\vec{J} = \rho_f\vec{v}_d = \underbrace{\frac{\rho_f q}{b}}_{\sigma}\vec{E}$$

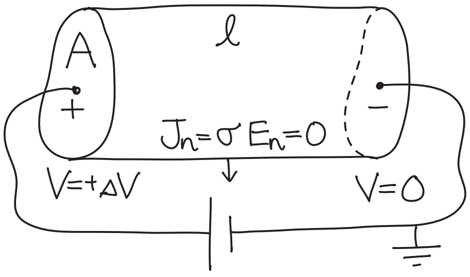
~ Drude law: bumper cars

$$V_d = \frac{\langle \frac{1}{2}at^2\rangle}{\langle t\rangle} = at = \frac{qE}{m}\cdot\frac{\lambda}{V_{rms}}$$

$$b = \frac{mV_{rms}}{\lambda} \qquad \sigma = \frac{(\rho_f q)}{b} = \frac{(nfq^2)\lambda}{m\,V_{rms}}$$

t = time between collisions
λ = mean free path
nf = atomic density x # carriers/atom

* RESISTOR - an electrical component



$$\nabla^2 V = 0 \qquad \text{B.C.'s} \Rightarrow V = \Delta V\cdot z/\ell$$

$$I = \vec{J}\cdot\vec{A} = \sigma EA = \sigma A/\ell\cdot\Delta V$$

$$= G\Delta V \qquad \boxed{G = \frac{\sigma A}{\ell}}$$ conductance
σ = conductivity

$$= \Delta V/R \qquad R = \frac{\rho\ell}{A}$$ resistance
ρ = resistivity

$$P = I\Delta V = I^2R = \frac{\Delta V^2}{R}$$ power dissipated

~ vs. CAPACITOR

$$Q = C\,\Delta V \qquad \boxed{C = \frac{\varepsilon A}{\ell}}$$ capacitance
ε = permittivity

$$U = \frac{1}{2}Q\Delta V = \frac{1}{2}\frac{Q^2}{\Delta V} = \frac{1}{2}C\Delta V^2$$

~ vs. INDUCTOR ... to be continued

* power dissipation

$$P = \vec{F}\cdot\vec{v}_d = q\vec{E}\cdot\vec{v}_d \qquad \dot{u} = \frac{du}{dt} = \frac{\Delta P}{\Delta\tau} = \rho_f v_d\cdot E = \vec{J}\cdot\vec{E} = \sigma E^2 = \rho J^2$$

$$u = \frac{\Delta W}{\Delta\tau} = \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{1}{2}\varepsilon E^2$$

* relaxation time

$$\frac{\partial\rho_f}{\partial t} = \nabla\cdot\vec{J} = \frac{\sigma}{\varepsilon}\nabla\cdot\vec{D} = \frac{\sigma}{\varepsilon}\rho_f(t) \qquad \rho = \rho_0 e^{-\sigma/\varepsilon t} \qquad \tau = \frac{\varepsilon}{\sigma} = RC$$

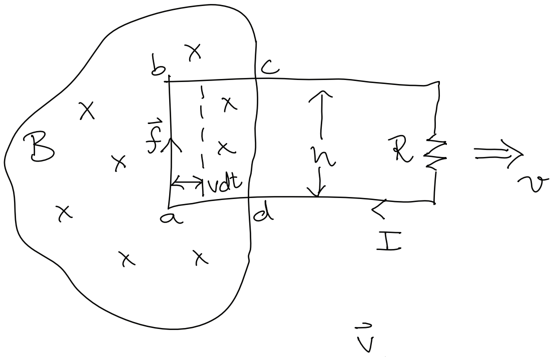
$$\tau = \frac{\varepsilon}{\sigma} = \frac{1/376.7\,c\cdot\Omega}{1/1.678\,\mu\Omega\cdot cm} = 0.445\,\text{Å}/c = 1.45\times10^{-19}s$$

* electromotive force (emf)

- ~ electromotance more correct! compare: magnetomotance (H/W4, #3)
- ~ forces on electrons from E and other sources (chemical, B, ...)
- ~ not quite $\mathcal{E}_E = \int \vec{E}\cdot d\vec{l}$ since $\mathcal{E}_E = 0$

$\vec{F} = q\vec{f}$ generalization of $\vec{E}$ $\qquad \mathcal{E} = \int \vec{f}\cdot d\vec{l}$ (emf)

* motional emf - magnetic forces



$$F = q(\vec{E} + v\times\vec{B}) \qquad \vec{f} = \vec{v}\times\vec{B}$$

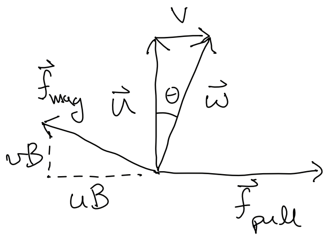
$$\mathcal{E} = \oint \vec{f}_{mag}\cdot d\vec{l} = vBh$$

~ relation to flux: precursor to Faraday's law

$$\Phi_B = \int \vec{B}\cdot d\vec{a} = Bhx$$

$$\frac{d\Phi}{dt} = Bh\frac{dx}{dt} = -Bhv = -\mathcal{E}$$

$$\boldsymbol{\mathcal{E} = -\frac{d\Phi}{dt}}$$



~ conservation of energy: magnetic force does no work!

$$\int \vec{f}_{pull}\cdot d\vec{l} = uB\frac{h}{\cos\theta}\sin\theta = uB\cdot h\cdot\frac{w}{u}\cdot\frac{v}{w} = Bhv = -\mathcal{E}$$

~ general proof

$$\mathcal{E} = \oint \vec{f}_{mag}\cdot d\vec{l}_s = \oint \vec{w}\times\vec{B}\cdot d\vec{l}_s$$

$$= -\oint \vec{B}\cdot(\vec{V} + \cancel{\vec{u}})\times d\vec{l}_s = \oint \vec{B}\cdot\frac{d\vec{l}_t\times d\vec{l}_s}{dt}$$

$$= \oint \vec{B}\cdot\frac{d\vec{a}}{dt} = -\frac{d\Phi}{dt}$$

net velocity:

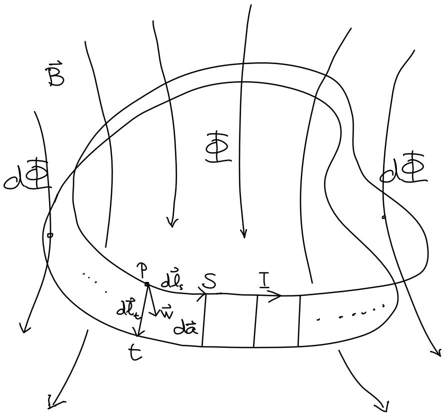
$\vec{w} = \vec{u} + \vec{v}$

along the wire:

$d\vec{l}_s = \vec{u}\,ds$

movement of wire:

$d\vec{l}_t = \vec{v}\,dt$



~ what about 'work' done by electromagnet lifting a car in the junkyard?

Section 7.2.1 - Faraday's Law

* three experiments - one result!
- a) moving loop in static B field (7.1)
- b) static loop in moving B field
- c) static loop in static changing B field

} change of reference frame (S.R.) (nonuniform field) ~ motional emf

} motion of flux lines irrelevant only net flux ~ Faraday's law

* different physics involved, both involving B fields

a) Lorentz force law
- moving charge in static field

$$\vec F = q(\vec E + \vec v\times\vec B)$$

$$\vec E_{eff} = \vec v\times\vec B$$

b,c) Faraday's law
- static charge in changing field

$$\mathcal{E}_E = \oint_{\partial S}\vec E\cdot d\vec l = \int_S \underbrace{\nabla\times\vec E}\cdot d\vec a = \int_S \underbrace{-\frac{\partial\vec B}{\partial t}}\cdot d\vec a = -\frac{d\Phi_B}{dt}$$

$$\nabla\times\vec E = -\frac{\partial\vec B}{\partial t}$$

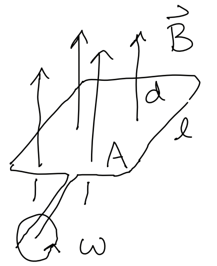
* Special Relativity
- ~ equivalence of E&M in different ref. frames
- ~ Lorentz transformations $\vec E \leftrightarrow \vec B$, both components of $\vec F = \vec E\,dt + \vec B$

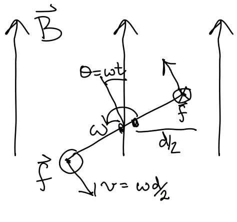
* Lenz's law
- ~ fields have "inertia"
- ~ it takes energy to build/destroy E,B
- ~ currents oppose change in fields

* Example of a) - AC generator

$$A(t) = A\cos(\omega t)$$

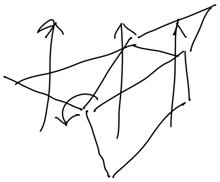
$$\mathcal{E} = -\frac{\partial\Phi}{\partial t} = AB\omega\sin(\omega t)$$

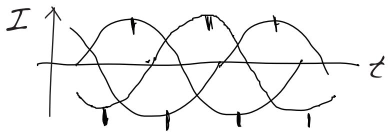




$$\mathcal{E} = \int\vec v\times\vec B = 2\cdot\int\frac{\omega d}{2}B\sin\theta\cdot dl$$
$$= AB\omega\sin(\omega t)$$

- ~ 3-phase generator has 6 maxima of current per cycle
- ~ both 1-phase and 2-phase only have 2 ~ bicycle pedal problem





* Example 7.5

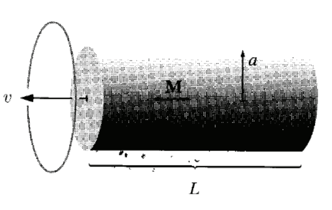


Figure 7.21

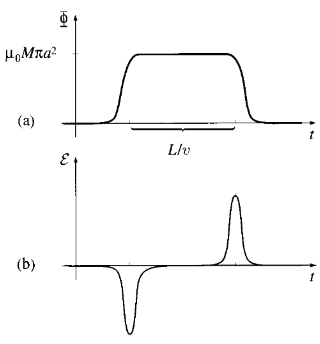


Figure 7.22

* Example 7.6

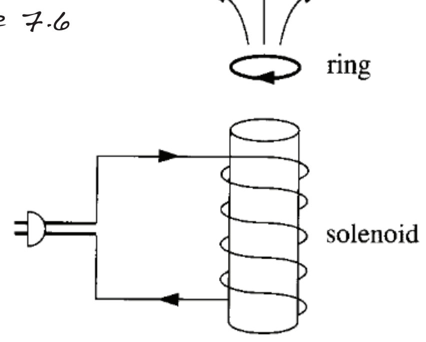


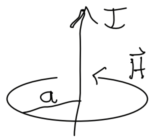
Figure 7.23

Section 7.2.2 - Induced Electric Field

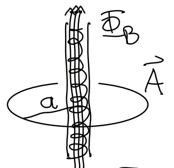
* three Ampere-like laws - one technique!

Ampere	Vector Potential		Faraday
$\nabla \times \vec{H} = \vec{J}$	$\nabla \times \vec{A} = \vec{B}$	$\longleftrightarrow$	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
$\mathcal{E}_H = \Phi_J = I$	$\mathcal{E}_A = \Phi_B$	solution to $\nabla \times E \neq 0$ no potential!	$\mathcal{E}_E = -\frac{d\Phi_B}{dt}$

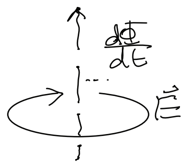
* with proper symmetry, each can be solved with Amperian loop



$$\vec{H} = \frac{I}{2\pi a}\hat{\phi}$$

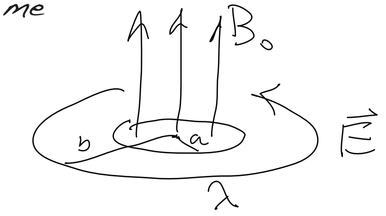


$$\vec{A} = \frac{\Phi_B}{2\pi a}\hat{\phi}$$



$$\vec{E} = -\frac{d\Phi_B}{2\pi a\, dt}$$

* Example 7.8: charge glued on a wheel
~ angular momentum from turning off field independent of time



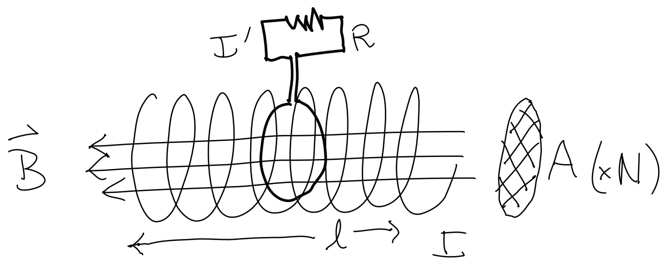
$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt} = -\pi a^2 \frac{dB}{dt}$$

$$dL = N\,dt = b\lambda \oint \vec{E}\cdot d\vec{l}\,dt = -b\lambda\,\pi a^2 \frac{dB}{\cancel{dt}}\cancel{dt}$$

~ alternate approach: vector potential (momentum)

$$d\vec{p} = \vec{F}dt = q\vec{E}dt = -q\frac{d\Phi}{2\pi a\,\cancel{dt}}\cancel{dt} = -q\,d\vec{A}$$

* Problem 7.12: mutual inductance



$n = N/l$

$$\Delta B_t = \mu_0 K_s = \mu_0 n I$$

$$\Phi = BA = \underbrace{\frac{\mu_0 A}{l}} NI \equiv \frac{1}{\mathcal{R}} NI \equiv \mathcal{P} NI$$

'reluctance' 'permeance'

$$I'R = \mathcal{E} = -\frac{d\Phi}{dt} = \mathcal{P}N\frac{dI}{dt}$$

Section 7.2.3 - Inductance

* review: 3 "Ampere" laws
~ will use all 3 today

$$\nabla \times \vec{H} = \vec{J} \qquad \nabla \times \vec{A} = \vec{B} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\mathcal{E}_H = \Phi_J = I \qquad \mathcal{E}_A = \Phi_B \qquad \mathcal{E}_E = -\frac{d\Phi_B}{dt}$$

* new Ohm's law
$V = I Z$

$$\frac{\text{current} \longrightarrow \text{flux}}{\text{time}} \longrightarrow \text{voltage}$$

$$I = I_0 e^{i\omega t}$$

$$V = \frac{1}{C}\int I dt = I \frac{1}{i\omega C}$$
$$V = IR \quad = IR$$
$$V = L\frac{dI}{dt} \quad = I i\omega L$$

* Mutual/Self Inductance - application of Faraday's law

$$B_1 = \frac{\mu_0 I_1}{4\pi}\oint_1 \frac{d\vec{l}_1 \times \hat{\mathscr{r}}}{\mathscr{r}^2} \qquad \Phi_2 = \int_2 \vec{B}_1 \cdot d\vec{a}_2$$

$$\Phi_2 = \int_2 \left(\frac{\mu_0}{4\pi}\oint_1 \frac{d\vec{l}_1 \times \hat{\mathscr{r}}}{\mathscr{r}^2}\right) \cdot d\vec{a}_2 \, I_1 \equiv M_{21} I_1$$

$$\Phi_2 = M_{21} I_1 \qquad \Phi = L I$$
$$\mathcal{E}_2 = -M_{21}\frac{dI_1}{dt} \qquad \mathcal{E} = -L\frac{dI}{dt}$$

$$M_{21} = \Phi_2 / I_1 = \mathcal{E}_A / I_1 = \oint_2 \underbrace{\left(\frac{\mu_0}{4\pi}\oint_1 \frac{d\vec{l}_1}{\mathscr{r}}\right)}_{\vec{A}/I_1} \cdot d\vec{l}_2 = \frac{\mu_0}{4\pi}\oint_1\oint_2 \frac{d\vec{l}_1 \cdot d\vec{l}_2}{\mathscr{r}}$$

~ property of material and geometry
~ "back" emf: voltage drop across L,
opposes changes in the current

compare: $\vec{F}_{21} = -\frac{\mu_0}{4\pi}\oint\oint \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} d\vec{l}_1 \cdot d\vec{l}_2 \; I_1 I_2$

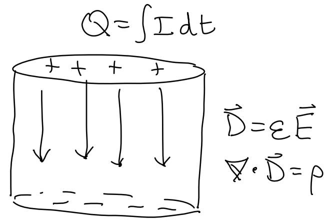
* Inductance matrix L
~ symmetric: mutual inductance
~ diagonal: self inductance

$$V_i = \sum_j L_{ij} \dot{I}_j$$
$$M_{21} = M_{12}$$
$$L_i \equiv M_{ii}$$

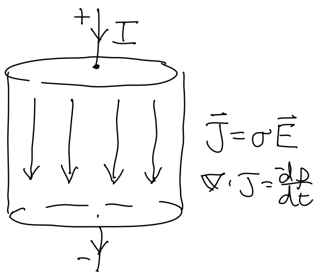
$$L = \begin{pmatrix} L_1 & M_{21} & M_{31} \\ M_{12} & L_2 & \ddots \\ M_{13} & \ddots & \ddots \end{pmatrix}$$

* three electrical devices - one calculation!

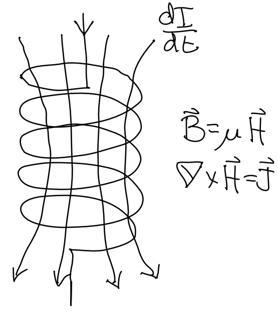
CAPACITOR



RESISTOR



INDUCTOR



$$Q = \sigma_f A = \int \vec{D} \cdot d\vec{a} = \Phi_D$$
$$V = \int \vec{E} \cdot d\vec{a} = \mathcal{E}_E$$
$$C = Q/V = \varepsilon \, \Phi/\mathcal{E} = \varepsilon\frac{A}{l}$$

$$I = \int \vec{J} \cdot d\vec{a} = \Phi_J$$
$$V = \text{same} = \mathcal{E}_E$$
$$R = V/I = \mathcal{E}/\sigma\Phi = \frac{l}{\sigma A}$$

$$N I = N \Phi_J = \mathcal{E}_H$$
$$V = -N\mathcal{E}_E = N\frac{d\Phi_B}{dt}$$
$$L = V/\dot{I} = \Phi/I = N^2 \mu \, \Phi/\mathcal{E} = N^2 \mu \frac{A}{l}$$

* units

$[C] = F$
$[\varepsilon_0] = F/m$

$[R] = \Omega$

$[L] = H$
$[\mu_0] = H/m$

Section 7.2.4 - Energy in the Magnetic Field

* *example: L-R circuit*

$$\mathcal{E} = IR + L\dot{I}$$

$$(\mathcal{E} - IR)\,dt = L\,dI = -(I - \mathcal{E}_0/R)\,dt$$

$$u\,dt = -L/R\,du$$

$$\ln(u/u_0) = \int \frac{du}{u} = -R/L \int dt$$

$$(\mathcal{E} - IR) = (\mathcal{E} - I_0 R)e^{-t/\tau}$$

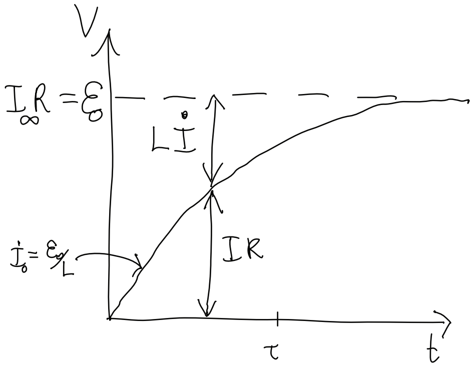
$$I = \frac{\mathcal{E}}{R} - \left(\frac{\mathcal{E}}{R} - I_0\right)e^{-t/\tau}$$

$$= I_\infty - \Delta I\, e^{-t/\tau}$$

let $u = \mathcal{E} - IR$
$du = -R\,dI$

let $\tau = L/R$
$u = u_0 e^{-t/\tau}$

where $\mathcal{E} = I_\infty R$
$\Delta I = I_\infty - I_0$



~ *time constant* $\tau = L/R$

note: initial slope depends on L, not R
larger R just means lower I_∞

* *work against back emf: "electrical inertia"*

$$\frac{dW}{dt} = -\mathcal{E}I = LI\frac{dI}{dt}$$

$$W = \tfrac{1}{2}I \oint \vec{A}\cdot d\vec{l}$$

$$= \frac{1}{2}\int_V \vec{A}\cdot\vec{J}\,d\tau$$

$$= \frac{1}{2\mu_0}\int_V \vec{A}\cdot\nabla\times\vec{B}\,d\tau$$

$$= \frac{1}{2\mu_0}\int_V B^2\,d\tau$$

$$\Delta W = \tfrac{1}{2}\int \Delta\vec{B}\cdot\vec{H}\,d\tau$$

$$W = \tfrac{1}{2}LI^2$$

$$LI = \Phi_B = \mathcal{E}_A$$

$$\tfrac{1}{2}\vec{p}\cdot\vec{v} = \tfrac{1}{2}mv^2$$ energy from "potential momentum"

$$\nabla\cdot(\vec{A}\times\vec{B}) = (\nabla\times\vec{A})\cdot\vec{B} - \vec{A}\cdot\nabla\times\vec{B}$$

$$\int_{\partial V}\vec{A}\times\vec{B}\cdot d\vec{a} = \int_V B^2\,d\tau - \int_V \vec{A}\cdot\nabla\times\vec{B}\,dt$$

compare:

$$\Delta W = \frac{1}{2}\int \Delta V\rho\,d\tau$$

$$= \frac{1}{2}\int \Delta V\cdot\nabla\cdot\vec{D}\,d\tau$$

$$= \frac{1}{2}\int -\Delta\nabla V\cdot\vec{D}\,d\tau$$

$$= \frac{1}{2}\int \Delta\vec{E}\cdot\vec{D}\,d\tau$$

$$u = \tfrac{1}{2}(E\cdot D + B\cdot H)$$

* *example 7.13*

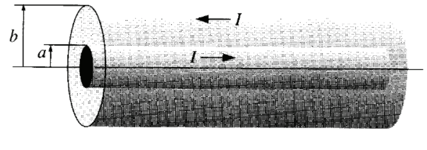


Figure 7.39

$$\vec{H} = \frac{I\hat{\phi}}{2\pi s} \qquad u = \tfrac{1}{2}\vec{B}\cdot\vec{H} = \frac{\mu I^2}{8\pi^2 s^2}$$

$$\frac{1}{2}\frac{LI^2}{l} = W/l = \int u\,da = \int_{s=a}^{b} 2\pi s\,ds\,\frac{\mu I^2}{8\pi^2 s^2} = \int_a^b \frac{\mu I^2}{4\pi}\frac{ds}{s} = \frac{\mu I^2}{4\pi}\ln\frac{b}{a}$$

$$L/l = \frac{\mu}{4\pi}\ln\left(\frac{b}{a}\right)$$

Section 7.3, 10.1 - Maxwell's Equations

* towards a consistent system of field equations

gauge potential fields sources

$$\begin{tikzcd}
\Lambda \arrow[r, "d"] & (V,\vec{A}) \arrow[r, "d"] & (\vec{E},\vec{B}) \arrow[r, "d"] \arrow[d, shift right, "\varepsilon"'] & 0 \\
dd=0 & & (\vec{D},\vec{H}) \arrow[r, "d", "\text{Maxwell}"'] \arrow[u, shift right, "\mu"'] & (\rho,\vec{J}) \arrow[r, "d", "\text{continuity}"'] & 0
\end{tikzcd}$$

* 2 problems

a) potentials

$$\nabla\times\vec{E} = -\frac{\partial B}{\partial t} \qquad \vec{E}\neq-\nabla V \quad\Longrightarrow\quad \nabla\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)=0$$

$$\nabla\cdot\vec{B}=0 \quad\Longrightarrow\quad \vec{B}=\nabla\times\vec{A} \qquad \vec{E}=-\nabla V-\partial_t\vec{A}$$

gauge invariance

$$\begin{array}{l} V\to V-\partial_t\Lambda \\ \vec{A}\to\vec{A}+\nabla\Lambda \end{array} \quad\Longrightarrow\quad \begin{array}{l} \vec{E}\to\vec{E}+(\cancel{\partial_t\Lambda}-\cancel{\partial_t\Lambda})=0 \\ \vec{B}\to\vec{B}+\cancel{\nabla\times(\nabla\Lambda)} \end{array}$$

$$\boxed{\begin{array}{ll}
V\to V-\partial_t\Lambda & \vec{A}\to\vec{A}+\nabla\Lambda \\
\vec{E}=-\nabla V-\partial_t\vec{A} & \vec{B}=\nabla\times\vec{A} \\
\nabla\times\vec{E}+\partial_t\vec{B}=0 & \nabla\cdot\vec{B}=0 \\
\nabla\times\vec{H}-\partial_t\vec{D}=\vec{J} & \nabla\cdot\vec{D}=\rho \\
\partial_t\rho+\nabla\cdot\vec{J}=0 & \vec{J}=\sigma\vec{E} \\
\vec{D}=\varepsilon\vec{E}=\varepsilon_0(\vec{E})+\vec{P} & \vec{B}=\mu\vec{H}=\mu_0(\vec{H}+\vec{M}) \\
\multicolumn{2}{l}{\vec{F}=q(\vec{E}+\vec{v}\times\vec{B})=(\rho\vec{E}+\vec{J}\times\vec{B})d\tau}
\end{array}}$$

b) continuity

$$\nabla\cdot(\nabla\times E)=-\frac{\partial}{\partial t}\nabla\cdot B=0$$

"displacement current"

$$\nabla\cdot(\nabla\times\vec{H})=\nabla\cdot\vec{J}\neq-\frac{\partial\rho}{\partial t}=-\frac{\partial}{\partial t}\nabla\cdot\vec{D} \quad\Longrightarrow\quad \nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$$

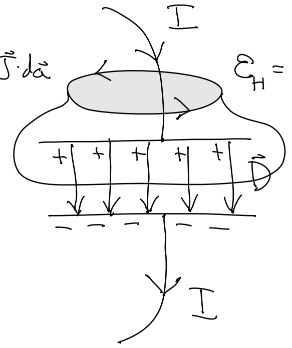
* example: capacitor - continuity: $I_{enc}=\frac{dQ}{dt}$

- Ampere's law should not depend on surface to integrate charge flux
- field should also exist in capacitor
- each new charge on plate builds up a new D-flux line $Q=\Phi_D$
- charge "propagates" through capacitor via its associate D-flux line $\Phi_J=I=\frac{dQ}{dt}=\frac{d\Phi_D}{dt}$
- "displacement current": I flowing through wire = D building up in capacitor

$I=\Phi_J=\int\vec{J}\cdot d\vec{a}$

$I=0?!$

$\mathcal{E}_H=\int_P\vec{H}\cdot d\vec{l}$



* expand D, H

$$\nabla\cdot(\varepsilon_0\vec{E}+\vec{P})=\rho_{tot}-\rho_b=\rho_f \qquad \vec{D}=\varepsilon_0\vec{E}+\vec{P} \qquad \nabla\cdot\vec{P}=-\rho_b$$

$$\nabla\cdot\vec{B}=0 \qquad \vec{B}=\mu_0(\vec{H}+\vec{M}) \qquad \nabla\times\vec{M}=J_b$$

$$\nabla\times\vec{E}+\partial_t B=0$$

$$\nabla\times\left(\frac{1}{\mu_0}\vec{B}-\vec{M}\right)-\partial_t(\varepsilon_0\vec{E}+\vec{P})=\vec{J}_f$$

"displacement current"

$$\vec{J}_{tot}=\vec{J}_b+\underbrace{(\vec{J}_{d,tot}-\vec{J}_p)}_{\vec{J}_d}+\vec{J}_f \qquad \vec{J}_d\equiv\frac{\partial\vec{D}}{\partial t}=\varepsilon\frac{\partial\vec{E}}{\partial t}$$

* Maxwell's Eq's in vacuum

$$\nabla \cdot \vec{E} = \rho/\varepsilon_0 \qquad \nabla \times \vec{E} + \partial_t \vec{B} = 0$$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} - \varepsilon_0 \mu_0 \partial_t \vec{E} = \mu_0 \vec{J}$$

* integral form

$$\Phi_D = Q \qquad \mathcal{E}_E + \partial_t \Phi_B = 0$$

$$\Phi_B = 0 \qquad \mathcal{E}_H - \partial_t \Phi_D = I$$

* boundary conditions - integrate Maxwell's equations over the surface

$$\nabla \to \hat{n}\Delta \qquad \rho \to \sigma \qquad \vec{J} \to \vec{K} \qquad \int_{-\varepsilon}^{\varepsilon} dn \frac{\partial}{\partial n} = \Delta \qquad \int_{-\varepsilon}^{\varepsilon} dn\, \delta(n) = 1$$

Fields		Integral		Potentials	
$\hat{n} \cdot \Delta \vec{D} = \sigma$	$\hat{n} \times \Delta \vec{E} = 0$	$\Delta \Phi_D = Q$	$\Delta \mathcal{E}_E = -\Delta V = 0$	$-\Delta\, \varepsilon \frac{\partial V}{\partial n} = \sigma$	$-\Delta \frac{\partial V}{\partial t} = 0$
$\hat{n} \cdot \Delta \vec{B} = 0$	$\hat{n} \times \Delta \vec{H} = \vec{K}$	$\Delta \Phi_B = 0$	$\Delta \mathcal{E}_H = -\Delta U = I$	$-\Delta\, \mu \frac{\partial U}{\partial n} = 0$	$-\Delta \frac{\partial U}{\partial t} = K_s$

* duality transformation - another symmetry of Maxwell's equations

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

- ~ without sources, B <=> E symmetry, except units
- ~ symmetry with sources by adding magnetic charge (monopole) $\rho_m, \vec{J}_m$
- ~ single magnetic monopole in universe would imply quantization of charge $\frac{\mu_0 q_e q_m}{4\pi} = \frac{n\hbar}{2}$
- ~ magnetic contributions can be "rotated away" as long as q_e/q_m is constant

$$\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho_e \qquad \nabla \times \vec{E} = -\mu_0 \vec{J}_m - \frac{\partial B}{\partial t}$$

$$\nabla \cdot \vec{B} = \mu_0 \rho_m \qquad \nabla \times \vec{B} = \mu_0 \vec{J}_e + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$-\mu_0 \left(\nabla \cdot \vec{J}_m + \frac{\partial \rho_m}{\partial t} \right) = 0 \quad \text{(continutity)}$$

$$\mu_0 \left(\nabla \cdot \vec{J}_e + \frac{\partial \rho_e}{\partial t} \right) = 0$$

Electromagnetism in a Nutshell

* Maxwell's equations et al.

gauge   potential   field   source

$$\begin{tikzcd}
\lambda \arrow[r, "d"] & (V,\vec{A}) \arrow[r, "d"] & (\vec{E},\vec{B}) \arrow[r, "d"] \arrow[d, "\mu"', "\varepsilon"] & 0 & \\
& U \arrow[r, "d"] & (\vec{D},\vec{H}) \arrow[r, "d"] & (\rho,\vec{J}) \arrow[r, "d"] & 0
\end{tikzcd}$$

invariance   Maxwell eq.'s

Poisson   continuity

$$V \to V - \partial_t \Lambda \qquad \vec{A} \to \vec{A} + \nabla\Lambda$$

$$\vec{E} = -\nabla V - \partial_t \vec{A} \qquad \vec{B} = \nabla \times \vec{A}$$

$$\nabla \times \vec{E} + \partial_t \vec{B} = 0 \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{H} - \partial_t \vec{D} = \vec{J} \qquad \nabla \cdot \vec{D} = \rho$$

$$\partial_t \rho + \nabla \cdot \vec{J} = 0 \qquad \vec{J} = \sigma \vec{E}$$

$$\vec{D} = \varepsilon \vec{E} = \varepsilon_0 (\vec{E}) + \vec{P} \qquad \vec{B} = \mu \vec{H} = \mu_0 (\vec{H} + \vec{M})$$

$$d\vec{F} = dq(\vec{E} + \vec{v} \times \vec{B}) = (\rho \vec{E} + \vec{J} \times \vec{B})\, d\tau$$

* Flux and Flow

~ conserved currents

$$\Phi_J,\ \Phi_S,\ \Phi_{\vec{T}}$$

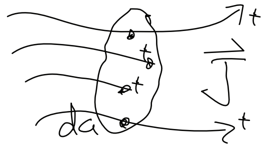
$$dq \sim \lambda dl \sim \sigma da \sim \rho d\tau$$

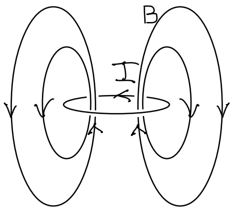
$$dq\,\vec{v} \sim I d\vec{l} \sim \vec{K} da \sim \vec{J} d\tau$$

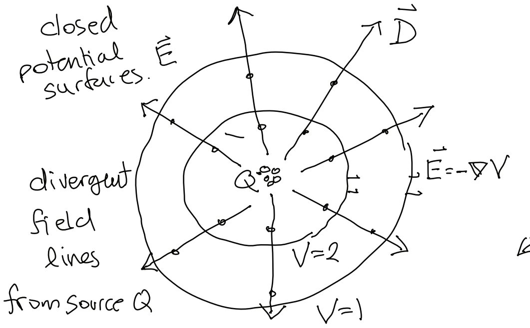
~ integral equations

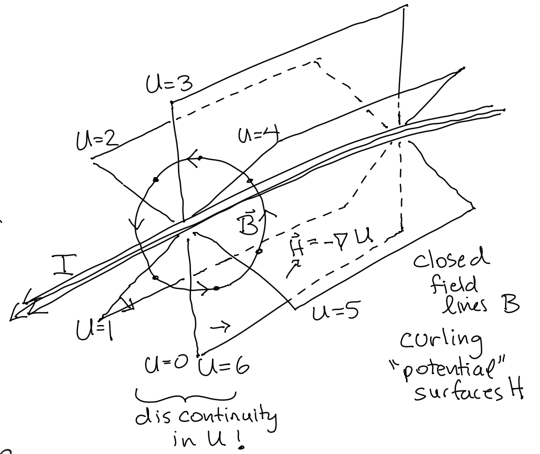
$$\Phi_D = Q \qquad \mathcal{E}_E + \partial_t \Phi_B = 0$$

$$\Phi_B = 0 \qquad \mathcal{E}_H - \partial_t \Phi_D = I$$









* utility of D & B flux lines, E & H equipotential surfaces

~ flux through a surface S = $\Phi_B = \int_S \vec{B} \cdot d\vec{a}$ = # of lines that poke through a surface S

~ flow along a curve/path P = $\mathcal{E}_E = \int_P E \cdot d\vec{l}$ = # of surfaces that a path P pokes through

* potentials, from Helmholtz theorem, qV = potential energy $q\vec{A}$ = "potential momentum"

~ transverse and longitudinal components $n^2 \vec{v} = \vec{n}\,\vec{n} \cdot \vec{v} - \vec{n} \times \vec{n} \times \vec{v}$ $\nabla = \hat{n} \frac{\partial}{\partial n} + \nabla_t$

$$\vec{E} = -\vec{\nabla}\underbrace{\Big(\nabla^{-2} \overbrace{\nabla \cdot \vec{E}}^{\rho}\Big)}_{V} + \nabla \times \Big(-\nabla^{-2} \overbrace{\nabla \times \vec{E}}^{}\Big)$$

$$= -\vec{\nabla} \; V$$

$$\vec{B} = -\vec{\nabla}\Big(\nabla^{-2} \overbrace{\nabla \cdot \vec{B}}^{0}\Big) + \nabla \times \underbrace{\Big(-\nabla^{-2} \overbrace{\nabla \times \vec{B}}^{\vec{J}}\Big)}_{\vec{A}}$$

$$\vec{B} = \nabla \times \vec{A}$$

* boundary conditions - integrate Maxwell's equations over the surface

$$\nabla \to \hat{n}\Delta \qquad \rho \to \sigma \qquad \vec{J} \to \vec{K} \qquad \int_{-\varepsilon}^{\varepsilon} dn \frac{\partial}{\partial n} = \Delta \qquad \int_{-\varepsilon}^{\varepsilon} dn\, \delta(n) = 1$$

~ electric

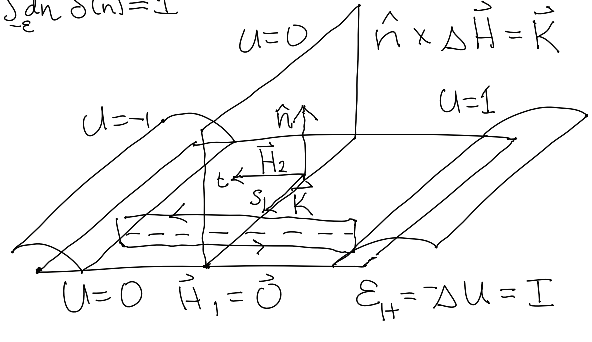
$$\mathcal{E}_E = 0 \qquad \Delta V = 0 \qquad E_{2t} = E_{1t} \qquad \hat{n} \times \Delta \vec{E} = 0$$

$$\Phi_D = Q \qquad -\Delta \varepsilon \frac{\partial V}{\partial n} = \sigma \qquad D_{2n} - D_{1n} = \sigma \qquad \hat{n} \cdot \Delta \vec{D} = \sigma$$

~ magnetic

$$\mathcal{E}_H = I \qquad -\Delta U = I \qquad H_{2t} - H_1 = K_s \qquad \hat{n} \times \Delta \vec{H} = \vec{K}$$

$$\Phi_B = 0 \qquad \Delta \mu \frac{\partial U}{\partial n} = 0 \qquad B_{2n} = B_{1n} \qquad \hat{n} \cdot \Delta \vec{B} = 0$$



~ surface current flows along U equipotential

~ U is a SOURCE potential

~ the current I=I2-I1 flows between any two equipotential lines U=I1 and U=I2

* electric magnetic dipoles and macroscopic equations (electric and magnetic materials)

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3} \qquad \vec{m} \equiv \oint I\, d\vec{a} = I\vec{a} \qquad \times \Leftrightarrow \cdot \qquad V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3} \qquad \frac{\vec{p}}{4\pi\varepsilon_0} \leftrightarrow \frac{\mu_0\vec{m}}{4\pi} \qquad \vec{M} \doteq \frac{1}{\tau}\int_v d\tau\, \vec{m}$$

* dynamics of dipoles in fields (compare Electric and Magnetic)

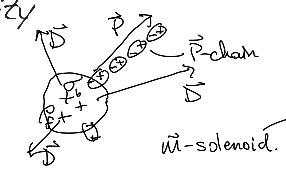
$$\vec{F}_e = (\vec{p}\cdot\nabla)\vec{E} \;=\; \nabla(\vec{p}\cdot\vec{E}) = -\nabla W \qquad (\nabla\times\vec{E}=0) \qquad \vec{N}_e = \vec{p}\times\vec{E} \qquad W = -\int N\, d\theta \;=\; -\vec{p}\cdot\vec{E}$$

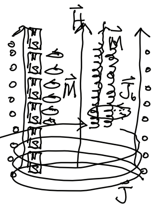
$$\vec{F}_m = (\vec{m}\times\nabla)\times\vec{B} \;=\; \nabla(\vec{m}\cdot\vec{B}) = -\nabla W \qquad (\nabla\cdot\vec{B}=0) \qquad \vec{N}_m = \vec{m}\times\vec{B} \qquad W = -\int N\, d\theta \;=\; -\vec{m}\cdot\vec{B}$$

* constitutive relations: magnetic susceptibility and permeability

$$\varepsilon_0\vec{E} = \vec{D} - \vec{P} \qquad \vec{D} = \varepsilon_0(\vec{E}+\vec{P}) \;=\; \varepsilon_0(1+\chi_e)\vec{E} = \varepsilon_0\varepsilon_r\vec{E} = \varepsilon\vec{E}$$

$$\frac{1}{\mu_0}\vec{B} = \vec{H} + \vec{M} \qquad \vec{B} = \mu_0(\vec{H}+\vec{M}) \;=\; \mu_0(1+\chi_m)\vec{H} = \mu_0\underbrace{\mu_r}_{K_m}\vec{H} = \mu\vec{H}$$



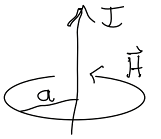


* three Ampere-like laws - each can be solved using Stoke's theorem

Ampere

$$\nabla\times\vec{H} = \vec{J}$$

$$\mathcal{E}_H = \Phi_J = I$$

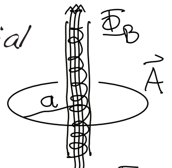


$$\vec{H} = \frac{I}{2\pi a}\hat{\phi}$$

Vector Potential

$$\nabla\times\vec{A} = \vec{B}$$

$$\mathcal{E}_A = \Phi_B$$

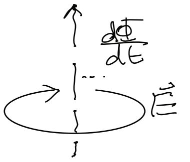


$$\vec{A} = \frac{\Phi_B}{2\pi a}\hat{\phi}$$

Faraday

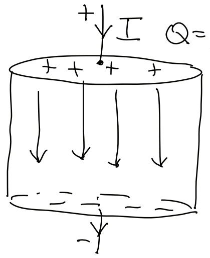
$$\nabla\times\vec{E} = \frac{\partial\vec{B}}{\partial t}$$

$$\mathcal{E}_E = \frac{-d\Phi_B}{dt}$$



$$\vec{E} = \frac{-d\Phi_B}{2\pi a\, dt}$$

* three passive electrical devices - each calculated by flux/flow = energy in field



$$Q = \int I\, dt \qquad \vec{D} = \varepsilon\vec{E}$$

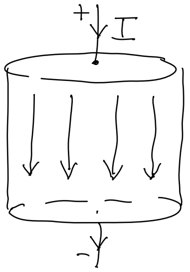
$$\nabla\cdot\vec{D} = \rho$$

$$Q = \sigma_f A = \int\vec{D}\cdot d\vec{a} = \Phi_D$$

$$V = \int\vec{E}\cdot d\vec{a} = \mathcal{E}_E$$

$$C = Q/V \;=\; \varepsilon\,\Phi/\mathcal{E} = \varepsilon\frac{A}{l}$$

$$[C] = F \qquad [\varepsilon_0] = F/m$$



$$\vec{J} = \sigma\vec{E}$$

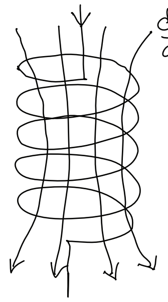
$$\nabla\cdot J = \frac{\partial\rho}{dt}$$

$$I = \int\vec{J}\cdot d\vec{a} = \Phi_J$$

$$V = \text{same} = \mathcal{E}_E$$

$$R = V/I = \mathcal{E}/\sigma\Phi = \frac{l}{\sigma A}$$

$$[R] = \Omega \qquad [\sigma] = m/\Omega$$



$$\vec{B} = \mu\vec{H}$$

$$\nabla\times\vec{H} = \vec{J}$$

$$NI = N\Phi_J = \mathcal{E}_H$$

$$V = -N\mathcal{E}_E = N\frac{d\Phi_B}{dt}$$

$$L = V/\dot{I} = \Phi/I = N^2\mu\Phi/\mathcal{E} = N^2\frac{\mu A}{l}$$

$$[L] = H \qquad [\mu_0] = H/m$$

* conserved currents

$$T_{\mu\nu} = \begin{array}{cccc} & \text{desity} & \text{flux} & \\ & u & \vec{S} & \text{energy} \\ & \vec{\wp} & \overleftrightarrow{T} & \text{momentum} \end{array}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = \partial_\mu J^\mu = 0$$

$$\frac{\partial}{\partial t}(u_{mech} + u_{em}) + \nabla\cdot\vec{S} = 0 \qquad \vec{S} \equiv \vec{E}\times\vec{H}$$

$$\frac{\partial\vec{\wp}}{\partial t} + \nabla\cdot\overleftrightarrow{T} = 0 \qquad \overleftrightarrow{T} \equiv (\vec{D}\vec{E} + \vec{B}\vec{H}) - \tfrac{1}{2}(\vec{D}\cdot\vec{E} + \vec{B}\cdot\vec{H})\overleftrightarrow{I}$$

* wave equations (Helmholtz) and solutions (Green's functions)

$$\left(\varepsilon\mu\frac{\partial^2}{\partial t^2} - \nabla^2\right)V = \rho/\varepsilon \qquad \left(\varepsilon\mu\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{A} = \mu\vec{J}$$

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r}',t_r)\,d\tau'}{\mathscr{r}} \qquad A(\vec{r},t) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}',t_r)\,d\tau}{\mathscr{r}}$$

$$\left(\varepsilon\mu\frac{\partial^2}{\partial t^2} - \nabla^2\right)\lambda = 0$$

$$\left(\varepsilon\mu\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{E} = 0$$

$$\left(\varepsilon\mu\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{B} = 0$$

* application of oblique boundary conditions: Fresnel equations

i) $D_{1n} = D_{2n}$:

ii) $B_{1n} = B_{2n}$:

iii) $E_{1t} = E_{2t}$:

iv) $H_{1t} = H_{2t}$:

$$E_I - E_R = \beta E_T \qquad E_R = \frac{\alpha-\beta}{\alpha+\beta}E_I \qquad R = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2$$

$$E_I + E_R = \alpha E_T \qquad E_T = \frac{2}{\alpha+\beta}E_I \qquad T = \frac{4\alpha\beta}{(\alpha+\beta)^2}$$

$$I = \frac{1}{2}\varepsilon v E^2\cos\theta.$$

$$= \frac{1}{2}\kappa\cos\theta\, E^2$$

$$\alpha = \frac{\cos\theta_2}{\cos\theta_1} \qquad \beta = \frac{\kappa_2}{\kappa_1}$$

* guides: wave equation for longitudinal component, boundary conditions

$$\left[\nabla_t^2 + \left(\omega/c\right)^2 - k^2\right]\begin{Bmatrix} E_z \\ B_z \end{Bmatrix} = 0.$$

$$(TE) \quad B_z(x,y)e^{ikz} \quad \left.\frac{\partial B_z}{\partial n}\right|_s = 0 \quad E_z = 0$$

$$(TM) \quad E_z(x,y)e^{ikz} \quad E_z\big|_s = 0 \quad B_z = 0$$