Arizona State University, Physics 311
Problem Set #2, Rev. B, due Wednesday, 2014-01-29

1. Coordinate basis

The *coordinate basis* $\boldsymbol{b}_i = \partial \boldsymbol{r}/\partial q^i$, *reciprocal basis* $\boldsymbol{b}^i = \boldsymbol{\nabla} q^i = \partial q^i/\partial \boldsymbol{r}$, and *unit basis* $\hat{e}_i = \boldsymbol{b}_i/h_i$ (where the *scale factor* is $h_i = |\boldsymbol{b}_i|$), are the natural bases to describe components of a vector field in a curvilinear coordinate system (q^1, q^2, q^3). Note that these basis vectors change from one point to the next. Calculate each of the following in both cylindrical and spherical coordinates.

a) Calculate $(\boldsymbol{b}_s, \boldsymbol{b}_\phi, \boldsymbol{b}_z)$ as a functions of $q^i = (s, \phi, z)$ and $(\boldsymbol{b}_r, \boldsymbol{b}_\theta, \boldsymbol{b}_\phi)$ as functions of (r, θ, ϕ).

b) Calculate the resulting scale factors h_i to get unit vectors. Write out the basis transformation matrix in terms of these vectors, i.e. $(\hat{\boldsymbol{s}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{z}}) = (\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})R$.

c) Construct the transformation matrices between unit bases, by considering rotations $R_{\hat{\boldsymbol{z}}}(\phi)$ (rotation about an angle ϕ about the z-axis) and $R_{\hat{\boldsymbol{\phi}}}(\theta)$ and compare with part b.

d) Calculate the reciprocal vectors $(\boldsymbol{b}^s, \boldsymbol{b}^\phi, \boldsymbol{b}^z)$ and $(\boldsymbol{b}^r, \boldsymbol{b}^\theta, \boldsymbol{b}^\phi)$. Show that $\boldsymbol{b}_i \cdot \boldsymbol{b}^j = \delta_i{}^j$ in general.

e) Calculate the line element $\boldsymbol{dl} = \boldsymbol{b}_i dq^i$, area element $\boldsymbol{da} = \frac{1}{2}\boldsymbol{dl} \times \boldsymbol{dl}$, and volume element $d\tau = \frac{1}{6}\boldsymbol{dl}\cdot\boldsymbol{dl}\times\boldsymbol{dl}$. Note that since differentials anticommute, their cross products are not zero.

f) Bonus: Calculate the metric $g_{ij} = \boldsymbol{b}_i \cdot \boldsymbol{b}_j$ and $g^{ij} = \boldsymbol{b}^i \cdot \boldsymbol{b}^j$ in terms of h_i. Show they are reciprocal.

g) Bonus: Calculate the derivatives of the basis vectors along coordinate lines, $\boldsymbol{\Gamma}_{ij} = \Gamma_{ij}{}^k \boldsymbol{b}_k = \partial \boldsymbol{b}_i/\partial q^j = \partial^2 \boldsymbol{r}/\partial q^i\, \partial q^j$. These *Christoffel symbols* and are needed to calculate derivatives of vectors in curvilinear coordinates.

Also, Griffiths 3ed[4ed] chapter 1, problems
#13[13], 19[20], 21[22], 25[26], 38[39], 39[40], 40[41], 42[43].