Arizona State University, Physics 311
Problem Set #5, Rev. A, due Wednesday, 2014-02-26

1. **Finite Element Method**. In this problem we will investigate one of the most common methods of solving partial differential equations numerically (especially involving the Laplacian). This method is very flexible and can be used to solve PDEs on irregularly shaped domains (like cars, or electrodes for your awesome new experiment). This problem is self-contained, but for extra details or hints, refer to the article http://wikipedia.org/wiki/Finite_element.

a) Show that

$$\int_R (\nabla^2 u) v d\tau = \oint_{\partial R} (\nabla u) v \cdot \boldsymbol{da} - \int_R \nabla u \cdot \nabla v d\tau. \tag{1}$$

Assuming that $v = 0$ on the boundary, this means that the equation $\nabla^2 u = f$ can be written

$$-\int_R \nabla u \cdot \nabla v d\tau = \int_R f v d\tau, \tag{2}$$

which is now a first order integral equation, which must be true for any *test function* $v(\boldsymbol{r})$.

b) In a one-dimensional space, Eq. 2 can be discretized by choosing appropriate "basis functions" for $v(\boldsymbol{r})$. Define the *tent function*

$$v_i(x) = (1 - |x - i|)\, \theta(1 - |x - i|), \tag{3}$$

where $\theta(x) = \{1 \text{ if } x > 0, \text{ and } 0 \text{ if } x < 0\}$ is the Heaviside step function and $i = 1, 2, 3, 4$. Plot each of these functions on the same graph.

c) Sketch the function $f(x) = 2v_1(x) + 4v_2(x) + 3v_3(x) + 1v_4(x)$. In the same way, any function defined on $0 < x < 5$ can be approximated by a linear combination of these basis functions $f(x) \approx \sum_{i=1}^{4} f_i v_i(x)$, where $f_i = f(i)$.

d) Convert Eq. 2 into a matrix equation by substituting $v(x) \to v_i(x)$ for $i = 1, 2, 3, 4$, approximating $u(x) = \sum_j u_j v_j(x)$ and $f(x) = \sum_j f_j v_j(x)$, and performing the integrals $\int_{-\infty}^{\infty} v_i(x) v_j(x) dx$ and $\int_{-\infty}^{\infty} \nabla v_i(x) \cdot \nabla v_j(x) dx$. Note that $\nabla = d/dx$ in one dimension.

e) Use part d) to solve the boundary value problem $\nabla^2 u = 3$ on the region $0 < x < 5$, with boundary conditions $u(0) = 0$ and $u(5) = 0$, by solving the above matrix equation for u_i.

f) Find the analytic solution $u(x)$ of part e) and compare with the finite element result.

g) Describe how this method could apply to higher dimensions. Hint: Define the 2-dimensional tent function $v_{ij}(x, y) = v_i(x)\, v_j(y)$. Plot this basis function, and evaluate at least two of the integrals that would be needed to form the matrix equation. We won't do a 2- or 3-dimensional problem, because the matrices become very large. However there are packages like FlexPDE (http://www.pdesolutions.com, free student version) and COMSOL (http://comsol.com) which do all the bookkeeping to implement this method for arbitrary geometries and complex differential equations, which you specify using a graphical user interface (GUI).

Also, Griffiths 3ed[4ed] Ch. 3, #1[1], 2[2], 5[5], 10[11]